

UNIQUE CONTINUATION PROPERTY AND LOCAL ASYMPTOTICS OF SOLUTIONS TO FRACTIONAL ELLIPTIC EQUATIONS

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ABSTRACT. Asymptotics of solutions to fractional elliptic equations with Hardy type potentials is studied in this paper. By using an Almgren type monotonicity formula, separation of variables, and blow-up arguments, we describe the exact behavior near the singularity of solutions to linear and semilinear fractional elliptic equations with a homogeneous singular potential related to the fractional Hardy inequality. As a consequence we obtain unique continuation property for fractional elliptic equations.

1. INTRODUCTION

The purpose of the present paper is to describe the asymptotic behavior of solutions to the following class of fractional elliptic semilinear equations with singular homogeneous potentials

$$(1) \quad (-\Delta)^s u(x) - \frac{\lambda}{|x|^{2s}} u(x) = h(x)u(x) + f(x, u(x)), \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain containing the origin,

$$(2) \quad N > 2s, \quad s \in (0, 1), \quad \lambda < \Lambda_{N,s} := 2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)},$$

$$(3) \quad h \in C^1(\Omega \setminus \{0\}), \quad |h(x)| + |x \cdot \nabla h(x)| \leq C_h |x|^{-2s+\varepsilon} \text{ as } |x| \rightarrow 0,$$

$$(4) \quad \begin{cases} f \in C^1(\Omega \times \mathbb{R}), \quad t \mapsto F(x, t) \in C^1(\Omega \times \mathbb{R}), \\ |f(x, t)t| + |f'_t(x, t)t^2| + |\nabla_x F(x, t) \cdot x| \leq C_f |t|^p \text{ for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}, \end{cases}$$

where $2 < p \leq 2^*(s) = \frac{2N}{N-2s}$, $F(x, t) = \int_0^t f(x, r) dr$, $C_f, C_h, \varepsilon > 0$ are positive constants independent of $x \in \Omega$ and $t \in \mathbb{R}$, $\nabla_x F$ denotes the gradient of F with respect to the x variable, and $f'_t(x, t) = \frac{\partial f}{\partial t}(x, t)$.

We recall that for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ and $s \in (0, 1)$, the fractional Laplacian $(-\Delta)^s \varphi$ is defined as

$$(5) \quad (-\Delta)^s \varphi(x) = C(N, s) \text{P.V.} \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N+2s}} dy = C(N, s) \lim_{\rho \rightarrow 0^+} \int_{|x-y|>\rho} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N+2s}} dy$$

where P.V. indicates that the integral is meant in the principal value sense and

$$C(N, s) = \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma(2-s)} s(1-s).$$

The Dirichlet form associated to $(-\Delta)^s$ on $C_c^\infty(\mathbb{R}^N)$ is given by

$$(6) \quad (u, v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \frac{C(N, s)}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy = \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{v}(\xi) \widehat{u}(\xi) d\xi,$$

where \widehat{u} denotes the unitary Fourier transform of u . It defines a scalar product thanks to (8) or (9) below. From now on we define $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion on $C_c^\infty(\mathbb{R}^N)$ with respect to the norm induced by the scalar product (6).

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By a weak solution to (1) we mean a function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ such that

$$(7) \quad (u, \varphi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_{\Omega} \left(\frac{\lambda}{|x|^{2s}} u(x) + h(x)u(x) + f(x, u(x)) \right) \varphi(x) dx, \text{ for all } \varphi \in C_c^\infty(\Omega).$$

We notice that the right hand side of (7) is well defined in view of assumptions (3)–(4), the Hardy-Littlewood-Sobolev inequality

$$(8) \quad S_{N,s} \|u\|_{L^{2^*(s)}(\mathbb{R}^N)}^2 \leq \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2,$$

and the following Hardy inequality, due to Herbst in [17] (see also [25]),

$$(9) \quad \Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{2s}} dx \leq \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u}^2(\xi) d\xi = \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2, \text{ for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

It should also be remarked that, in (7), we allow $p = 2^*(s)$ and that u is not prescribed outside Ω .

One of the aim of this paper is to give the precise behavior of a solution u to (1). The rate and the shape of u are given by the the eigenvalues and the eigenfunctions of the following eigenvalue problem

$$(10) \quad \begin{cases} -\operatorname{div}_{\mathbb{S}^N}(\theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi) = \mu \theta_1^{1-2s} \psi, & \text{in } \mathbb{S}_+^N, \\ -\lim_{\theta_1 \rightarrow 0^+} \theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi \cdot \mathbf{e}_1 = \kappa_s \lambda \psi, & \text{on } \partial \mathbb{S}_+^N, \end{cases}$$

where

$$(11) \quad \kappa_s = \frac{\Gamma(1-s)}{2^{2s-1} \Gamma(s)}$$

and

$$\mathbb{S}_+^N = \{(\theta_1, \theta_2, \dots, \theta_{N+1}) \in \mathbb{S}^N : \theta_1 > 0\} = \left\{ \frac{z}{|z|} : z \in \mathbb{R}^{N+1}, z \cdot \mathbf{e}_1 > 0 \right\},$$

with $\mathbf{e}_1 = (1, 0, \dots, 0)$; we refer to section 2.1 for a variational formulation of (10). From classical spectral theory (see section 2.1 for the details) problem (10) admits a diverging sequence of real eigenvalues with finite multiplicity

$$\mu_1(\lambda) \leq \mu_2(\lambda) \leq \dots \leq \mu_k(\lambda) \leq \dots$$

Moreover $\mu_1(\lambda) > -\left(\frac{N-2s}{2}\right)^2$, see Lemma 2.2.

Our first main result is the following theorem.

Theorem 1.1. *Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a nontrivial solution to (1) in a bounded domain $\Omega \subset \mathbb{R}^N$ containing the origin as in (7) with s, λ, h and f satisfying assumptions (2), (3) and (4). Then there exists an eigenvalue $\mu_{k_0}(\lambda)$ of (10) and an eigenfunction ψ associated to $\mu_{k_0}(\lambda)$ such that*

$$(12) \quad \tau^{-\frac{2s-N}{2} - \sqrt{\left(\frac{2s-N}{2}\right)^2 + \mu_{k_0}(\lambda)}} u(\tau\theta') \rightarrow \psi(0, \theta') \quad \text{in } C^{1,\alpha}(\mathbb{S}^{N-1}) \quad \text{as } \tau \rightarrow 0^+,$$

for some $\alpha \in (0, 1)$, where $\mathbb{S}^{N-1} = \partial \mathbb{S}_+^N$.

We should mention that the above result is stated in such form for the sake of simplicity. In fact, for the eigenfunction ψ in (12), we obtain precisely its components in any basis of the eigenspace corresponding to $\mu_{k_0}(\lambda)$, see Theorem 4.1 (below). We also remark that $\mu_1(\lambda) < 0$ for $\lambda > 0$ and it is determined implicitly by the usual Gamma function, see Proposition 2.3.

Our next result contains the so called *strong unique continuation property* which is a direct consequence of Theorem 1.1.

Theorem 1.2. *Suppose that all the assumptions of Theorem 1.1 hold true. Let u be a solution to (1) in a bounded domain $\Omega \subset \mathbb{R}^N$ containing the origin. If $u(x) = O(|x|^n)$ as $|x| \rightarrow 0$ for all $n \in \mathbb{N}$, then $u \equiv 0$ in Ω .*

Recent research in the field of second order elliptic equations has devoted a great attention to the problem of unique continuation property in the presence of singular lower order terms, see e.g. [10, 18, 20]. Two different kinds of approach have been developed to treat unique continuation: a first one is due to Carleman [5] and is based on weighted priori inequalities, whereas a second one is due to Garofalo and Lin [15] and is based on local doubling properties proved by Almgren monotonicity formula. In the present paper we will follow the latter approach. Furthermore, in the

spirit of [11, 12, 13], the combination of monotonicity methods with blow-up analysis will enable us to prove not only unique continuation but also the precise asymptotics of solutions stated in Theorem 1.1.

To explain our argument let us write (7) as

$$(13) \quad (-\Delta)^s u = G(x, u) \quad \text{in } \Omega.$$

The proof of our results is based on the study of the *Almgren frequency function* at the origin 0: “ratio of the local energy over mass near the origin”. Due to the non-locality of the Dirichlet form associated to $(-\Delta)^s$, it is not clear how to set up an Almgren’s type frequency function using this energy as in the local case $s = 1$. A way out for this difficulty is to use the Caffarelli-Silvestre extension [4] which can be seen as a local version of (13). The Caffarelli-Silvestre extension of a solution u to (13) is a function w defined on

$$\mathbb{R}_+^{N+1} = \{z = (t, x) : t \in (0, +\infty), x \in \mathbb{R}^N\}$$

satisfying $w = u$ on Ω and solving in some weak sense (see Section 2 for more details) the boundary value problem

$$(14) \quad \begin{cases} \operatorname{div}(t^{1-2s} \nabla w) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ -\lim_{t \rightarrow 0^+} t^{1-2s} \frac{\partial w}{\partial t} = \kappa_s G(x, w), & \text{on } \Omega. \end{cases}$$

Here and in the following, we write $z = (t, x) \in \mathbb{R}_+^{N+1}$ with $x \in \mathbb{R}^N$ and $t > 0$, and we identify \mathbb{R}^N with $\partial \mathbb{R}_+^{N+1}$, so that Ω is contained in $\partial \mathbb{R}_+^{N+1}$.

We then consider the Almgren’s frequency function

$$\mathcal{N}(r) = \frac{D(r)}{H(r)}$$

where

$$D(r) = \frac{1}{r^{N-2s}} \left[\int_{B_r^+} t^{1-2s} |\nabla w|^2 dt dx - \kappa_s \int_{B_r^+} G(x, w) w dx \right], \quad H(r) = \frac{1}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} w^2 dS,$$

being

$$\begin{aligned} B_r^+ &= \{z = (t, x) \in \mathbb{R}_+^{N+1} : |z| < r\}, & B_r' &:= \{x \in \mathbb{R}^N : |x| < r\}, \\ S_r^+ &= \{z = (t, x) \in \mathbb{R}_+^{N+1} : |z| = r\}, \end{aligned}$$

and dS denoting the volume element on N -dimensional spheres.

As a first but nontrivial step, we prove that $\lim_{r \rightarrow 0} \mathcal{N}(r) := \gamma$ exists and it is finite, see Lemma 3.14. Next, we make a blow-up analysis by zooming around the origin the solution w normalized also by \sqrt{H} . More precisely, setting $w_\tau(z) = \frac{w(\tau z)}{\sqrt{H(\tau)}}$, we have that w_τ converges, (in some Hölder and Sobolev spaces) to \tilde{w} solving the limiting equation

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla \tilde{w}) = 0, & \text{in } B_1^+, \\ -\lim_{t \rightarrow 0^+} t^{1-2s} \frac{\partial \tilde{w}}{\partial t} = \frac{\kappa_s \lambda}{|z|^{2s}} \tilde{w}, & \text{on } B_1'. \end{cases}$$

To obtain this, the fact that h is negligible with respect to the Hardy potential and f is at most critical with respect to the Sobolev exponent (see assumptions (3) and (4)) plays a crucial role; we refer to Lemma 4.2 for more details.

The main point is that the Almgren’s frequency for \tilde{w} is $\tilde{\mathcal{N}}(r) = \lim_{\tau \rightarrow 0} \mathcal{N}(\tau r) = \gamma$, i.e. $\tilde{\mathcal{N}}$ is constant; hence \tilde{w} and $\nabla \tilde{w} \cdot \frac{z}{|z|}$ are proportional on $L^2(S_r^+; t^{1-2s})$. As a consequence we obtain $\tilde{w}(z) = \varphi(|z|) \psi(\frac{z}{|z|})$. By separating variables in polar coordinates, we obtain that ψ is an eigenfunction of (10) for some eigenvalue $\mu_{k_0}(\lambda)$. By the method of variation of constants and the fact that \tilde{w} has finite energy near the origin, we then prove that $\varphi(r)$ is proportional to r^γ and that $\gamma = \frac{2s-N}{2} + \sqrt{\left(\frac{2s-N}{2}\right)^2 + \mu_{k_0}(\lambda)}$. We finally complete the proof by showing that $\lim_{r \rightarrow 0} r^{-2\gamma} H(r) > 0$, see Lemma 4.5.

We should mention that in the recent literature, a great attention has been addressed to non-local fractional diffusion and many papers have been devoted to the study of existence, non-existence, regularity and qualitative properties of solutions to elliptic equations associated to fractional Laplace type operators, see e.g. [2, 3, 4, 6, 8, 9, 22, 23] and references therein.

2. PRELIMINARIES AND NOTATIONS

Notation. We list below some notation used throughout the paper.

- $\mathbb{S}^N = \{z \in \mathbb{R}^{N+1} : |z| = 1\}$ is the unit N -dimensional sphere.
- $\mathbb{S}_+^N = \{(\theta_1, \theta_2, \dots, \theta_{N+1}) \in \mathbb{S}^N : \theta_1 > 0\} := \mathbb{S}^N \cap \mathbb{R}_+^{N+1}$.
- dS denotes the volume element on N -dimensional spheres.
- dS' denotes the volume element on $(N-1)$ -dimensional spheres.

Let $\mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ be the completion of $C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$ with respect to the norm

$$\|w\|_{\mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})} = \left(\int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla w(t, x)|^2 dt dx \right)^{1/2}.$$

We recall that there exists a well defined continuous trace map $\text{Tr} : \mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s}) \rightarrow \mathcal{D}^{s,2}(\mathbb{R}^N)$, see e.g. [2].

For every $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, let $\mathcal{H}(u) \in \mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ be the unique solution to the minimization problem

$$(15) \quad \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla \mathcal{H}(u)|^2 dt dx = \min \left\{ \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla v|^2 dt dx : v \in \mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s}), \text{Tr}(v) = u \right\}.$$

By Caffarelli and Silvestre [4] we have that

$$(16) \quad \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla \mathcal{H}(u) \cdot \nabla \tilde{\varphi} dt dx = \kappa_s(u, \text{Tr} \tilde{\varphi})_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{for all } \tilde{\varphi} \in \mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s}),$$

where κ_s is defined in (11).

We notice that combining (9), (15), (16) and (8), we obtain the following Hardy-trace inequality

$$(17) \quad \kappa_s \Lambda_{N,s} \int_{\mathbb{R}^N} \frac{(\text{Tr } v)^2(x)}{|x|^{2s}} dx \leq \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla v|^2 dt dx, \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$$

and also the Sobolev-trace inequality

$$(18) \quad \kappa_s S_{N,s} \|\text{Tr } v\|_{L^{2^*(s)}(\mathbb{R}^N)}^2 \leq \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla v|^2 dt dx, \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s}).$$

2.1. Separation of variables in the extension operator. For every $R > 0$, we define the space $H^1(B_R^+; t^{1-2s})$ as the completion of $C^\infty(\overline{B_R^+})$ with respect to the norm

$$\|w\|_{H^1(B_R^+; t^{1-2s})} = \left(\int_{B_R^+} t^{1-2s} (|\nabla w(t, x)|^2 + w^2(t, x)) dt dx \right)^{1/2}.$$

By direct calculations, we can obtain the following lemma concerning separation of variables in the extension operator.

Lemma 2.1. *If $v \in H^1(B_R^+; t^{1-2s})$ is such that $v(z) = f(r)\psi(\theta)$ for a.e. $z = (t, x) \in \mathbb{R}_+^{N+1}$, with $r = |z| < R$ and $\theta = \frac{z}{|z|} \in \mathbb{S}^N$, then*

$$\text{div}(t^{1-2s} \nabla v(z)) = \frac{1}{r^N} (r^{N+1-2s} f')' \theta_1^{1-2s} \psi(\theta) + r^{-1-2s} f(r) \text{div}_{\mathbb{S}^N} (\theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi(\theta))$$

in the distributional sense, where $\theta_1 = \frac{t}{r} = \theta \cdot \mathbf{e}_1$ with $\mathbf{e}_1 = (1, 0, \dots, 0)$, $\text{div}_{\mathbb{S}^N}$ (respectively $\nabla_{\mathbb{S}^N}$) denotes the Riemannian divergence (respectively gradient) on the unit sphere \mathbb{S}^N endowed with the standard metric.

Let us define $H^1(\mathbb{S}_+^N; \theta_1^{1-2s})$ as the completion of $C^\infty(\overline{\mathbb{S}_+^N})$ with respect to the norm

$$\|\psi\|_{H^1(\mathbb{S}_+^N; \theta_1^{1-2s})} = \left(\int_{\mathbb{S}_+^N} \theta_1^{1-2s} (|\nabla_{\mathbb{S}^N} \psi(\theta)|^2 + \psi^2(\theta)) dS \right)^{1/2}.$$

We also denote

$$L^2(\mathbb{S}_+^N; \theta_1^{1-2s}) := \left\{ \psi : \mathbb{S}_+^N \rightarrow \mathbb{R} \text{ measurable such that } \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \psi^2(\theta) dS < +\infty \right\}.$$

The following trace inequality on the unit half-sphere \mathbb{S}_+^N holds.

Lemma 2.2. *There exists a well defined continuous trace operator*

$$H^1(\mathbb{S}_+^N; \theta_1^{1-2s}) \rightarrow L^2(\partial\mathbb{S}_+^N) = L^2(\mathbb{S}^{N-1}).$$

Moreover for every $\psi \in H^1(\mathbb{S}_+^N; \theta_1^{1-2s})$

$$\kappa_s \Lambda_{N,s} \left(\int_{\mathbb{S}^{N-1}} |\psi(\theta')|^2 dS' \right) \leq \left(\frac{N-2s}{2} \right)^2 \int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\psi(\theta)|^2 dS + \int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\nabla_{\mathbb{S}^N} \psi(\theta)|^2 dS.$$

where dS' denotes the volume element on the sphere $\mathbb{S}^{N-1} = \partial\mathbb{S}_+^N = \{(\theta_1, \theta') \in \mathbb{S}_+^N : \theta_1 = 0\}$.

PROOF. Let $\psi \in C^\infty(\overline{\mathbb{S}_+^N})$ and $f \in C_c^\infty(0, +\infty)$ with $f \neq 0$. Rewriting (17) for $v(z) = f(r)\psi(\theta)$, $r = |z|$, $\theta = \frac{z}{|z|}$, we obtain that

$$\begin{aligned} \kappa_s \Lambda_{N,s} & \left(\int_0^{+\infty} r^{N-1-2s} f^2(r) dr \right) \left(\int_{\mathbb{S}^{N-1}} |\psi(0, \theta')|^2 dS' \right) \\ & \leq \left(\int_0^{+\infty} r^{N+1-2s} |f'(r)|^2 dr \right) \left(\int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\psi(\theta)|^2 dS \right) \\ & \quad + \left(\int_0^{+\infty} r^{N-1-2s} f^2(r) dr \right) \left(\int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\nabla_{\mathbb{S}_+^N} \psi(\theta)|^2 dS \right), \end{aligned}$$

and hence, by optimality of the classical Hardy constant, see [16],

$$\begin{aligned} \kappa_s \Lambda_{N,s} & \left(\int_{\mathbb{S}^{N-1}} |\psi(0, \theta')|^2 dS' \right) \\ & \leq \left(\int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\psi(\theta)|^2 dS \right) \inf_{f \in C_c^\infty(0, +\infty)} \frac{\int_0^{+\infty} r^{N+1-2s} |f'(r)|^2 dr}{\int_0^{+\infty} r^{N-1-2s} f^2(r) dr} + \int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\nabla_{\mathbb{S}_+^N} \psi(\theta)|^2 dS \\ & = \left(\frac{N-2s}{2} \right)^2 \int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\psi(\theta)|^2 dS + \int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\nabla_{\mathbb{S}_+^N} \psi(\theta)|^2 dS. \end{aligned}$$

By density of $C^\infty(\overline{\mathbb{S}_+^N})$ in $H^1(\mathbb{S}_+^N; \theta_1^{1-2s})$, we obtain the conclusion. ■

In view of Lemma 2.1, in order to construct an orthonormal basis of $L^2(\mathbb{S}_+^N; \theta_1^{1-2s})$ for expanding solutions to (14) in Fourier series, we are naturally lead to consider the eigenvalue problem (10), which admits the following variational formulation: we say that $\mu \in \mathbb{R}$ is an eigenvalue of problem (10) if there exists $\psi \in H^1(\mathbb{S}_+^N; \theta_1^{1-2s}) \setminus \{0\}$ (called eigenfunction) such that

$$Q(\psi, v) = \mu \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \psi(\theta) v(\theta) dS, \quad \text{for all } v \in H^1(\mathbb{S}_+^N; \theta_1^{1-2s}),$$

where

$$\begin{aligned} Q & : H^1(\mathbb{S}_+^N; \theta_1^{1-2s}) \times H^1(\mathbb{S}_+^N; \theta_1^{1-2s}) \rightarrow \mathbb{R}, \\ Q(\psi, v) & = \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi(\theta) \cdot \nabla v(\theta) dS - \lambda \kappa_s \int_{\mathbb{S}^{N-1}} \mathcal{T} \psi(\theta') \mathcal{T} v(\theta') dS'. \end{aligned}$$

By Lemma 2.2 the bilinear form Q is continuous and weakly coercive on $H^1(\mathbb{S}_+^N; \theta_1^{1-2s})$. Moreover the belonging of the weight t^{1-2s} to the second Mackenhaupt class ensures that the embedding $H^1(\mathbb{S}_+^N; \theta_1^{1-2s}) \hookrightarrow L^2(\mathbb{S}_+^N; \theta_1^{1-2s})$ is compact (see [7] for weighted embeddings with Mackenhaupt weights). Then, from classical spectral theory (see e.g. [21, Theorem 6.16]), problem (10) admits

a diverging sequence of real eigenvalues with finite multiplicity $\mu_1(\lambda) \leq \mu_2(\lambda) \leq \dots \leq \mu_k(\lambda) \leq \dots$ the first of which admits the variational characterization

$$(19) \quad \mu_1(\lambda) = \min_{\psi \in H^1(\mathbb{S}_+^N; \theta_1^{1-2s}) \setminus \{0\}} \frac{Q(\psi, \psi)}{\int_{\mathbb{S}_+^N} \theta_1^{1-2s} \psi^2(\theta) dS}.$$

Furthermore, in view of Lemma 2.2, we have that

$$(20) \quad \mu_1(\lambda) > -\left(\frac{N-2s}{2}\right)^2.$$

To each $k \geq 1$, we associate an $L^2(\mathbb{S}_+^N; \theta_1^{1-2s})$ -normalized eigenfunction $\psi_k \in H^1(\mathbb{S}_+^N; \theta_1^{1-2s}) \setminus \{0\}$ corresponding to the k -th eigenvalue $\mu_k(\lambda)$, i.e. satisfying

$$(21) \quad Q(\psi_k, v) = \mu_k(\lambda) \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \psi_k(\theta) v(\theta) dS, \quad \text{for all } v \in H^1(\mathbb{S}_+^N; \theta_1^{1-2s}).$$

In the enumeration $\mu_1(\lambda) \leq \mu_2(\lambda) \leq \dots \leq \mu_k(\lambda) \leq \dots$ we repeat each eigenvalue as many times as its multiplicity; thus exactly one eigenfunction ψ_k corresponds to each index $k \in \mathbb{N}$, $k \geq 1$. We can choose the functions ψ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}_+^N; \theta_1^{1-2s})$.

We can also determine $\mu_1(\lambda)$ for $\lambda \in (0, \Lambda_{N,s})$, where $\Lambda_{N,s}$ is the fractional Hardy constant defined in (2).

Proposition 2.3. *For every $\alpha \in (0, \frac{N-2s}{2})$, we define*

$$\lambda(\alpha) = 2^{2s} \frac{\Gamma\left(\frac{N+2s+2\alpha}{4}\right) \Gamma\left(\frac{N+2s-2\alpha}{4}\right)}{\Gamma\left(\frac{N-2s-2\alpha}{4}\right) \Gamma\left(\frac{N-2s+2\alpha}{4}\right)}.$$

Then the mapping $\alpha \mapsto \lambda(\alpha)$ is continuous and decreasing. In addition we have that

$$\mu_1(\lambda(\alpha)) = \alpha^2 - \left(\frac{N-2s}{2}\right)^2 \quad \text{for all } \alpha \in \left(0, \frac{N-2s}{2}\right).$$

PROOF. It was proved in [8, Lemma 3.1] that, for every $\alpha \in (0, \frac{N-2s}{2})$, there exists a positive continuous function $\Phi_\alpha : \mathbb{R}_+^{N+1} \rightarrow \mathbb{R}$ such that

$$(22) \quad \begin{cases} \operatorname{div}(t^{1-2s} \nabla \Phi_\alpha) = 0 & \text{in } \mathbb{R}_+^{N+1} \\ \Phi_\alpha = |x|^{\frac{2s-N}{2}+\alpha} & \text{on } \mathbb{R}^N \setminus \{0\} \\ -t^{1-2s} \frac{\partial \Phi_\alpha}{\partial t} = \kappa_s \lambda(\alpha) |x|^{-2s} \Phi_\alpha & \text{on } \mathbb{R}^N \setminus \{0\}, \end{cases}$$

Moreover $\Phi_\alpha \in H^1(B_R^+; t^{1-2s})$ for every $R > 0$ and Φ_α is scale invariant, i.e.

$$\Phi_\alpha(\tau z) = \tau^{\frac{2s-N}{2}+\alpha} \Phi_\alpha(z), \quad \text{for all } \tau > 0,$$

thus implying that, for all $z \in \mathbb{R}_+^{N+1}$,

$$(23) \quad c_1 |z|^{\frac{2s-N}{2}+\alpha} \leq \Phi_\alpha(z) \leq c_2 |z|^{\frac{2s-N}{2}+\alpha},$$

for some positive constants c_1, c_2 . It is also known (see for instance [14]) that the map $\alpha \mapsto \lambda(\alpha)$ is continuous and monotone decreasing.

We write

$$\Phi_\alpha(z) = \sum_{k=1}^{\infty} \Phi_\alpha^k(|z|) \psi_k(z/|z|), \quad \Phi_\alpha^k(r) = \int_{\mathbb{S}_+^N} \psi_k(\theta) \Phi_\alpha(r\theta) dS.$$

In particular, since $\psi_1 > 0$, by (23) we have, for every $r \in (0, R)$,

$$(24) \quad c'_1 |r|^{\frac{2s-N}{2}+\alpha} \leq \Phi_\alpha^1(r) \leq c'_2 |r|^{\frac{2s-N}{2}+\alpha},$$

for some positive constants c'_1, c'_2 . Using (22), we have, weakly, for every $k \geq 1$ and $r \in (0, R)$,

$$\begin{cases} \frac{1}{r^N} (r^{N+1-2s} (\Phi_\alpha^k)')' \theta_1^{1-2s} \psi_k(\theta) + r^{-1-2s} \Phi_\alpha^k \operatorname{div}_{\mathbb{S}^N} (\theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi_k(\theta)) = 0, \\ -\Phi_\alpha^k \lim_{\theta_1 \rightarrow 0^+} \theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi_k(\theta) \cdot \mathbf{e}_1 = \kappa_s \lambda(\alpha) \psi_k(0, \theta') \Phi_\alpha^k. \end{cases}$$

Testing the above equation with $\psi_1 > 0$ and using also the fact that $\Phi_\alpha^1 > 0$, we obtain

$$(\Phi_\alpha^1)'' + \frac{N+1-2s}{r} (\Phi_\alpha^1)' - \frac{\mu_1(\lambda(\alpha))}{r^2} \Phi_\alpha^1 = 0$$

and hence $\Phi_\alpha^1(r)$ is of the form

$$\Phi_\alpha^1(r) = c_3 r^{\sigma^+} + c_4 r^{\sigma^-}$$

for some $c_3, c_4 \in \mathbb{R}$, where

$$\sigma^+ = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(\lambda(\alpha))} \quad \text{and} \quad \sigma^- = -\frac{N-2s}{2} - \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(\lambda(\alpha))}.$$

Since the function $|z|^{\sigma_{k_0}^-} \psi_1\left(\frac{z}{|z|}\right) \notin H^1(B_1^+; t^{1-2s})$, we deduce that $c_4 = 0$ and thus for every $r \in (0, R)$

$$\Phi_1^\alpha(r) = c_3 r^{\sigma^+}.$$

This together with (24) implies that

$$\mu_1(\lambda(\alpha)) = \alpha^2 - \left(\frac{N-2s}{2}\right)^2 \quad \text{for all } \alpha \in \left(0, \frac{N-2s}{2}\right),$$

as claimed. ■

2.2. Hardy type inequalities. From well-known weighted embedding inequalities and the fact that the weight t^{1-2s} belongs to the second Mackenhaupt class (see e.g. [7]), the embedding $H^1(B_r^+; t^{1-2s}) \hookrightarrow L^2(B_r^+; t^{1-2s})$ is compact. It can be also proved that both the trace operators

$$(25) \quad H^1(B_r^+; t^{1-2s}) \hookrightarrow L^2(S_r^+; \theta_1^{1-2s}),$$

$$(26) \quad H^1(B_r^+; t^{1-2s}) \hookrightarrow L^2(B_r')$$

are well defined and compact.

For sake of simplicity, in the following of this paper, we will often denote the trace of a function with the same letter as the function itself.

The following Hardy type inequality with boundary terms holds.

Lemma 2.4. *For all $r > 0$ and $w \in H^1(B_r^+; t^{1-2s})$, the following inequality holds*

$$\left(\frac{N-2s}{2}\right)^2 \int_{B_r^+} t^{1-2s} \frac{w^2(z)}{|z|^2} dz \leq \int_{B_r^+} t^{1-2s} \left(\nabla w(z) \cdot \frac{z}{|z|}\right)^2 dz + \left(\frac{N-2s}{2r}\right) \int_{S_r^+} t^{1-2s} w^2 dS.$$

PROOF. By scaling, it is enough to prove the stated inequality for $r = 1$. Let

$$V(z) = |z|^{\frac{2s-N}{2}}, \quad z \in \mathbb{R}_+^{N+1} \setminus \{0\}.$$

We notice that V satisfies

$$(27) \quad -\operatorname{div}(t^{1-2s} \nabla V) = \left(\frac{N-2s}{2}\right)^2 t^{1-2s} |z|^{-2} V(z) \quad \text{in } \mathbb{R}_+^{N+1} \setminus \{0\}.$$

Hence, letting $w \in C^\infty(\overline{B_1^+})$, multiplying (27) with $\frac{w^2}{V}$, and integrating over $B_1^+ \setminus B_\delta^+$ with $\delta \in (0, 1)$, we obtain

$$\begin{aligned} & \left(\frac{N-2s}{2}\right)^2 \int_{B_1^+} t^{1-2s} \frac{w^2(z)}{|z|^2} dz \\ &= \int_{B_1^+ \setminus B_\delta^+} t^{1-2s} \nabla V(z) \cdot \nabla \left(\frac{w^2}{V}\right)(z) dz - \int_{S_1^+} t^{1-2s} (\nabla V \cdot \nu) \frac{w^2}{V} dS + \int_{S_\delta^+} t^{1-2s} (\nabla V \cdot \nu) \frac{w^2}{V} dS \\ &= -(N-2s) \int_{B_1^+ \setminus B_\delta^+} t^{1-2s} \frac{w}{|z|} \left(\nabla w \cdot \frac{z}{|z|}\right) dz - \left(\frac{N-2s}{2}\right)^2 \int_{B_1^+ \setminus B_\delta^+} t^{1-2s} \frac{w^2}{|z|^2} dz \\ & \quad + \frac{N-2s}{2} \int_{S_1^+} t^{1-2s} w^2 dS - \frac{N-2s}{2} \frac{1}{\delta} \int_{S_\delta^+} t^{1-2s} w^2 dS \end{aligned}$$

where $\nu(z) = \frac{z}{|z|}$. Since, by Schwarz's inequality

$$(28) \quad - (N-2s) \int_{B_1^+ \setminus B_\delta^+} t^{1-2s} \frac{w}{|z|} \left(\nabla w \cdot \frac{z}{|z|} \right) dz \\ \leq \left(\frac{N-2s}{2} \right)^2 \int_{B_1^+ \setminus B_\delta^+} t^{1-2s} \frac{w^2}{|z|^2} dz + \int_{B_1^+ \setminus B_\delta^+} t^{1-2s} \left(\nabla w \cdot \frac{z}{|z|} \right)^2 dz$$

and

$$\frac{1}{\delta} \int_{S_\delta^+} t^{1-2s} w^2 dS = O(\delta^{N-2s}) = o(1) \quad \text{for } \delta \rightarrow 0^+,$$

letting $\delta \rightarrow 0$ we obtain the stated inequality for $r = 1$ and $w \in C^\infty(\overline{B_1^+})$. The conclusion follows by density of $C^\infty(\overline{B_1^+})$ in $H^1(B_1^+; t^{1-2s})$. ■

Lemma 2.5. *For every $r > 0$ and $w \in H^1(B_r^+; t^{1-2s})$, the following inequality holds*

$$\kappa_s \Lambda_{N,s} \int_{B_r^+} \frac{w^2}{|x|^{2s}} dx \leq \left(\frac{N-2s}{2r} \right) \int_{S_r^+} t^{1-2s} w^2 dS + \int_{B_r^+} t^{1-2s} |\nabla w|^2 dz.$$

PROOF. Let $w \in C^\infty(\overline{B_r^+})$. Then, passing to polar coordinates and using Lemmas 2.2 and 2.4, we obtain

$$\begin{aligned} \kappa_s \Lambda_{N,s} \int_{B_r^+} \frac{w^2(0,x)}{|x|^{2s}} dx &= \kappa_s \Lambda_{N,s} \int_0^r \rho^{N-1-2s} \left(\int_{\mathbb{S}^{N-1}} w^2(0, \rho \theta') dS' \right) d\rho \\ &\leq \int_0^r \rho^{N-1-2s} \left(\left(\frac{N-2s}{2} \right)^2 \int_{\mathbb{S}_+^N} \theta_1^{1-2s} |w(\rho \theta)|^2 dS + \int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\nabla_{\mathbb{S}^N} w(\rho \theta)|^2 dS \right) d\rho \\ &= \left(\frac{N-2s}{2} \right)^2 \int_{B_r^+} t^{1-2s} \frac{w^2}{|z|^2} dz + \int_0^r \rho^{N-1-2s} \left(\int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\nabla_{\mathbb{S}^N} w(\rho \theta)|^2 dS \right) d\rho \\ &\leq \left(\frac{N-2s}{2r} \right) \int_{S_r^+} t^{1-2s} w^2 dS \\ &\quad + \int_0^r \rho^{N+1-2s} \left(\int_{\mathbb{S}_+^N} \theta_1^{1-2s} \left(\frac{1}{\rho^2} |\nabla_{\mathbb{S}^N} w(\rho \theta)|^2 + \left| \frac{\partial w}{\partial \rho}(\rho \theta) \right|^2 \right) dS \right) d\rho \\ &= \left(\frac{N-2s}{2r} \right) \int_{S_r^+} t^{1-2s} w^2 dS + \int_{B_r^+} t^{1-2s} |\nabla w|^2 dz. \end{aligned}$$

The conclusion follows by density of $C^\infty(\overline{B_r^+})$ in $H^1(B_r^+; t^{1-2s})$. ■

The following Sobolev type inequality with boundary terms holds.

Lemma 2.6. *There exists $\tilde{S}_{N,s} > 0$ such that, for all $r > 0$ and $w \in H^1(B_r^+; t^{1-2s})$,*

$$\left(\int_{B_r^+} |w|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \leq \tilde{S}_{N,s} \left[\frac{N-2s}{2r} \int_{S_r^+} t^{1-2s} w^2 dS + \int_{B_r^+} t^{1-2s} |\nabla w|^2 dz \right].$$

PROOF. By scaling, it is enough to prove the statement for $r = 1$. Let $w \in C^\infty(\overline{B_1^+})$ and denote by \tilde{w} its fractional Kelvin transform defined as $\tilde{w}(z) = |z|^{-(N-2s)} w(\frac{z}{|z|^2})$. Some computations (see also [9]) show that

$$(29) \quad \int_{B_1^+} t^{1-2s} |\nabla w|^2 dz + (N-2s) \int_{S_1^+} t^{1-2s} w^2 dS = \int_{\mathbb{R}_+^{N+1} \setminus B_1^+} t^{1-2s} |\nabla \tilde{w}|^2 dz,$$

$$(30) \quad \int_{B_1^+} |w|^{2^*(s)} dx = \int_{\mathbb{R}_+^{N+1} \setminus B_1^+} |\tilde{w}|^{2^*(s)} dx.$$

In particular the function

$$v(z) = \begin{cases} w(z), & \text{if } z \in B_1^+, \\ \tilde{w}(z), & \text{if } z \in \mathbb{R}_+^{N+1} \setminus B_1^+ \end{cases}$$

belongs to $\mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$, so that, by (18) we have

$$(31) \quad \left(\int_{\mathbb{R}^N} |v|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \leq \frac{1}{S_{N,s}\kappa_s} \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla v|^2 dz.$$

From (29), (30), and (31), it follows that

$$\left(2 \int_{B'_1} |w|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \leq \frac{2}{S_{N,s}\kappa_s} \left[\int_{B_1^+} t^{1-2s} |\nabla w|^2 dz + \frac{N-2s}{2} \int_{S_1^+} t^{1-2s} w^2 dS \right]$$

which, by density, yields the conclusion. ■

Combining Lemma 2.5 and Lemma 2.6 the following corollary follows.

Corollary 2.7. *For all $r > 0$ and $w \in H^1(B_r^+; t^{1-2s})$, the following inequalities hold*

$$(32) \quad \int_{B_r^+} t^{1-2s} |\nabla w|^2 dz - \kappa_s \lambda \int_{B'_r} \frac{w^2}{|x|^{2s}} dx + \frac{N-2s}{2r} \int_{S_r^+} t^{1-2s} w^2 dS \\ \geq \kappa_s (\Lambda_{N,s} - \lambda) \int_{B'_r} \frac{w^2}{|x|^{2s}} dx$$

and

$$(33) \quad \int_{B_r^+} t^{1-2s} |\nabla w|^2 dz - \kappa_s \lambda \int_{B'_r} \frac{w^2}{|x|^{2s}} dx + \frac{N-2s}{2r} \int_{S_r^+} t^{1-2s} w^2 dS \\ \geq \frac{\Lambda_{N,s} - \lambda}{(1 + \Lambda_{N,s}) \widetilde{S}_{N,s}} \left(\int_{B'_r} |w|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}}.$$

3. THE ALMGREN TYPE FREQUENCY FUNCTION

Let $R > 0$ be such that $B'_R \subset \subset \Omega$ and $w \in H^1(B_R^+; t^{1-2s})$ be a nontrivial solution to

$$(34) \quad \begin{cases} \operatorname{div}(t^{1-2s} \nabla w) = 0, & \text{in } B_R^+, \\ -\lim_{t \rightarrow 0^+} t^{1-2s} \frac{\partial w}{\partial t}(t, x) = \kappa_s \left(\frac{\lambda}{|x|^{2s}} w + hw + f(x, w) \right), & \text{on } B'_R, \end{cases}$$

in a weak sense, i.e., for all $\varphi \in C_c^\infty(B_R^+ \cup B'_R)$, we have that

$$(35) \quad \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla w \cdot \nabla \varphi dt dx = \kappa_s \int_{B'_R} \left(\frac{\lambda}{|x|^{2s}} w + hw + f(x, w) \right) \varphi dx,$$

with s, λ, h, f as in assumptions (2), (3), and (4).

The main result of this section is the existence of the limit as $r \rightarrow 0^+$ of the *Almgren's frequency* function (see [15] and [1]) associated to w

$$(36) \quad \mathcal{N}(r) = \frac{r \left[\int_{B_r^+} t^{1-2s} |\nabla w|^2 dt dx - \kappa_s \int_{B'_r} \left(\frac{\lambda}{|x|^{2s}} w^2 + h(x) w^2 + f(x, w) w \right) dx \right]}{\int_{S_r^+} t^{1-2s} w^2 dS}.$$

We notice that, by Lemma 2.5, $w(0, \cdot) \in L^2(B'_R; |x|^{-2s})$ and so the $L^1(0, R)$ -function

$$r \mapsto \int_{S_r^+} t^{1-2s} |\nabla w|^2 dS, \quad \text{respectively} \quad r \mapsto \int_{\partial B'_r} \frac{w^2}{|x|^{2s}} dS',$$

is the weak derivative of the $W^{1,1}(0, R)$ -function

$$r \mapsto \int_{B_r^+} t^{1-2s} |\nabla w|^2 dz, \quad \text{respectively} \quad r \mapsto \int_{B'_r} \frac{w^2}{|x|^{2s}} dx.$$

In particular, for a.e. $r \in (0, R)$, $\frac{\partial w}{\partial \nu} \in L^2(S_r^+; t^{1-2s})$, where $\nu = \nu(z)$ is the unit outer normal vector $\nu(z) = \frac{z}{|z|}$.

Lemma 3.1. *For a.e. $r \in (0, R)$ and every $\tilde{\varphi} \in C^\infty(\overline{B_r^+})$*

$$\int_{B_r^+} t^{1-2s} \nabla w \cdot \nabla \tilde{\varphi} dz = \int_{S_r^+} t^{1-2s} \frac{\partial w}{\partial \nu} \tilde{\varphi} dS + \kappa_s \int_{B_r'} \left(\frac{\lambda}{|x|^{2s}} w + hw + f(x, w) \right) \tilde{\varphi} dx.$$

PROOF. It follows by testing (35) with $\tilde{\varphi}(z)\eta_n(|z|)$ where $\eta_n(\rho) = 1$ if $\rho < r - \frac{1}{n}$, $\eta_n(r) = 0$ if $\rho > r$, $\eta(\rho) = n(r - \rho)$ is $r - \frac{1}{n} \leq \rho \leq r$, passing to the limit, and noticing that a.e. $r \in (0, R)$ is a Lebesgue point for the $L^1(0, R)$ -function $r \mapsto \int_{S_r^+} t^{1-2s} \frac{\partial w}{\partial \nu} \tilde{\varphi} dS$. ■

Lemma 3.2. (i) *Let $r > 0$ and $V \in L^q(B_r')$ for some $q > \frac{N}{2s}$. For every $t_0, r_0 > 0$ such that $[0, t_0] \times B_{r_0}' \subseteq B_r^+ \cup B_r'$ there exist positive constants $A_1 > 0$, $\alpha \in (0, 1)$ depending on $t_0, r_0, r, \|V\|_{L^q(B_r')}$ such that for every $v \in H^1(B_r^+; t^{1-2s})$ solving*

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla v) = 0, & \text{in } B_r^+ \\ -\lim_{t \rightarrow 0^+} t^{1-2s} v_t = V(x)v, & \text{on } B_r', \end{cases}$$

we have that $v \in C^{0,\alpha}([0, t_0] \times B_{r_0}')$ and

$$(37) \quad \|v\|_{C^{0,\alpha}([0, t_0] \times B_{r_0}')} \leq A_1 \|v\|_{H^1(B_r^+; t^{1-2s})}.$$

(ii) *Let $v \in H^1(B_r^+; t^{1-2s}) \cap C^{0,\alpha}(B_r^+)$ for some $\alpha \in (0, 1)$ be a function satisfying*

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla v) = 0, & \text{in } B_r^+ \\ -\lim_{t \rightarrow 0^+} t^{1-2s} v_t = g(x, v), & \text{on } B_r', \end{cases}$$

where $g \in C^1(B_r' \times \mathbb{R})$ and $|g(x, \rho)| \leq c(|\rho| + |\rho|^{p-1})$ for some $2 < p \leq 2^(s) = \frac{2N}{N-2s}$, $c > 0$, and every $x \in B_r'$ and $\rho \in \mathbb{R}$. Let $t_0, r_0 > 0$ such that $[0, t_0] \times B_{r_0}' \subseteq B_r^+$. Then there exist positive constants A_2, β depending only on $N, p, s, c, r, r_0, t_0, \|v\|_{H^1(B_r^+; t^{1-2s})}$ and $\|g\|_{C^1(B_{r_0}' \times [0, A_3])}$ where $A_3 = \|v\|_{C^{0,\alpha}([0, t_0] \times B_{r_0}')}^{\beta}$, with $\beta \in (0, 1)$, such that*

$$(38) \quad \|\nabla_x v\|_{C^{0,\beta}([0, t_0] \times B_{r_0}')} \leq A_2,$$

$$(39) \quad \|t^{1-2s} v_t\|_{C^{0,\beta}([0, t_0] \times B_{r_0}')} \leq A_2.$$

Remark 3.3. The dependence of the constant A_2 in Lemma 3.2 on $\|v\|_{H^1(B_r^+; t^{1-2s})}$ is continuous; in particular we can take the same A_2 for a family of solutions which are uniformly bounded in $H^1(B_r^+; t^{1-2s}) \cap C^{0,\alpha}(B_r^+)$.

PROOF. Part (i) and (37) follows from [[19], Proposition 2.4].

To prove (ii), for $h \in \mathbb{R}^N$ with $|h| < 1$, we set $v^h(t, x) = \frac{v(t, x+h) - v(t, x)}{|h|}$ for every $x \in B_{r_0/2}'$. Then we have

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla v^h) = 0, & \text{in } B_{r_0/2}^+ \\ -\lim_{t \rightarrow 0^+} t^{1-2s} v_t^h = c_h(x)v^h + b_h, & \text{on } B_{r_0/2}', \end{cases}$$

where

$$c_h(x) = \frac{g(x, v(0, x+h)) - g(x, v(0, x))}{v(0, x+h) - v(0, x)} \chi_{\{v(0, x+h) \neq v(0, x)\}}(x)$$

with χ_A being the characteristic function of a set A and

$$b_h(x) = \frac{g(x+h, v(0, x+h)) - g(x, v(0, x+h))}{|h|}.$$

Let $A_3 = \|v\|_{C^{0,\alpha}([0, t_0] \times B_{r_0}')}^{\beta}$. Then we have

$$\|c_h\|_{L^\infty(B_{r_0/4}')} + \|b_h\|_{L^\infty(B_{r_0/4}')} \leq \|g_\rho\|_{L^\infty(B_{r_0/2}' \times [0, A_3])} + \|\nabla_x g\|_{L^\infty(B_{r_0/2}' \times [0, A_3])},$$

for every small h . Applying once again [[19], Proposition 2.4],

$$\begin{aligned} \|v^h\|_{C^{0,\alpha}([0, t_0/8] \times B_{r_0/8}')} &\leq C(\|v^h\|_{L^\infty([0, t_0/4] \times B_{r_0/4}')} + \|b_h\|_{L^\infty(B_{r_0/2}')}) \\ &\leq C\|v^h\|_{L^2([0, t_0/2] \times B_{r_0/2}'; t^{1-2s})} + C\|\nabla g\|_{L^\infty(B_{r_0/2}' \times [0, A_3])} \\ &\leq C\|\nabla v\|_{L^2([0, t_0] \times B_{r_0}'; t^{1-2s})} + C\|\nabla g\|_{L^\infty(B_{r_0/2}' \times [0, A_3])} \\ &\leq A_2 \end{aligned}$$

for every small h . By the Arzelà-Ascoli Theorem, passing to the limit as $|h| \rightarrow 0$, we conclude that $z \mapsto \nabla_x v(z) \in C([0, t_0/8] \times B'_{r_0/8})$ and estimate (38) holds for all $\beta \in (0, \alpha)$.

Since the map $x \mapsto g(x, v(0, x)) \in C^{0,\beta}(B'_{r_0/8})$, estimate (39) follows from [[3], Lemma 4.5]. ■

Lemma 3.4. *Let w be a solution to (34) in the sense of (35). Then there exist $p_0 > 2^*(s)$ and $R_0 \in (0, R)$ such that $w \in L^{p_0}(B_{R_0}^+)$.*

PROOF. By (3), there are $\delta > 0$ and $R_\delta \in (0, R)$ such that

$$(40) \quad (\lambda + |x|^{2s}|h(x)|) \leq \lambda + \delta < \Lambda_{N,s}, \quad \text{for all } x \in B'_{R_\delta}.$$

Let $\beta > 1$. For all $L > 0$, we define $F_L(\tau) = |\tau|^\beta$ if $|\tau| < L$ and $F_L(\tau) = \beta L^{\beta-1}|\tau| + (1-\beta)L^\beta$ if $|\tau| \geq L$. Put $G = \frac{1}{\beta}F_L F'_L$. It is easy to verify that, for all $\tau \in \mathbb{R}$,

$$(41) \quad \tau G_L(\tau) \leq \tau^2 G'_L(\tau), \quad \tau G_L(\tau) \leq (F_L(\tau))^2, \quad (F'_L(\tau))^2 \leq \beta G'_L(\tau).$$

Let $\eta \in C_c^\infty(B_{R_\delta}^+ \cup B'_{R_\delta})$ be a radial cut-off function such that $\eta \equiv 1$ in $B_{R_\delta}^+$. It is clear that $\zeta := \eta^2 G_L(w) \in H^1(B_{R_\delta}^+; t^{1-2s})$ and $F_L(w) \in H^1(B_{R_\delta}^+; t^{1-2s})$. Using ζ as a test in (34), from (40) and integration by parts, we have that

$$\begin{aligned} & \int_{B_{R_\delta}^+} t^{1-2s} \eta^2 |\nabla w|^2 G'_L(w) dt dx - \kappa_s(\lambda + \delta) \int_{B'_{R_\delta}} |x|^{-2s} \eta^2 w G_L(w) dx \\ & \leq -2 \int_{B_{R_\delta}^+} t^{1-2s} \eta \nabla w \cdot \nabla \eta G_L(w) dt dx \\ & \quad + c \int_{B_{R_\delta}^+} \eta^2 w G_L(w) dx + c \int_{B'_{R_\delta}} \eta^2 |w|^{2^*(s)-2} w G_L(w) dx, \end{aligned}$$

for some positive $c > 0$ depending only on C_f, s, p, N . By Young's inequality and (41), we have that, for every $\sigma > 0$,

$$\left| 2(\eta \nabla w) \cdot (w \nabla \eta) \frac{G_L(w)}{w} \right| \leq \frac{\sigma}{2} \eta^2 |\nabla w|^2 G'_L(w) + \frac{2}{\sigma} |\nabla \eta|^2 (F_L(w))^2,$$

hence we obtain that

$$\begin{aligned} & \left(1 - \frac{\sigma}{2}\right) \int_{B_{R_\delta}^+} t^{1-2s} \eta^2 |\nabla w|^2 G'_L(w) dt dx - \kappa_s(\lambda + \delta) \int_{B'_{R_\delta}} |x|^{-2s} \eta^2 w G_L(w) dx \\ & \leq c \int_{B'_{R_\delta}} \eta^2 (|w|^{2^*(s)-2} + 1) w G_L(w) dx + \frac{2}{\sigma} \int_{B_{R_\delta}^+} t^{1-2s} |\nabla \eta|^2 \psi^2 dt dx, \end{aligned}$$

where we have set $\psi = F_L(w)$. Therefore by (41)

$$\begin{aligned} & \frac{1}{\beta} \left(1 - \frac{\sigma}{2}\right) \int_{B_{R_\delta}^+} t^{1-2s} \eta^2 |\nabla \psi|^2 dt dx - \kappa_s(\lambda + \delta) \int_{B'_{R_\delta}} |x|^{-2s} (\eta \psi)^2 dx \\ & \leq c \int_{B_{R_\delta}^+} (|w|^{2^*(s)-2} + 1) (\eta \psi)^2 dx + \frac{2}{\sigma} \int_{B_{R_\delta}^+} t^{1-2s} |\nabla \eta|^2 \psi^2 dt dx. \end{aligned}$$

Since $|\nabla(\eta \psi)|^2 \leq (1+\sigma)\eta^2 |\nabla \psi|^2 + (1+\frac{1}{\sigma})\psi^2 |\nabla \eta|^2$, we have that

$$\begin{aligned} & \frac{1}{\beta(1+\sigma)} \left(1 - \frac{\sigma}{2}\right) \int_{B_{R_\delta}^+} t^{1-2s} |\nabla(\eta \psi)|^2 dt dx - \kappa_s(\lambda + \delta) \int_{B'_{R_\delta}} |x|^{-2s} (\eta \psi)^2 dx \\ & \leq c \int_{B_{R_\delta}^+} (|w|^{2^*(s)-2} + 1) (\eta \psi)^2 dx + C(c, \beta, \sigma, R_\delta) \int_{B_{R_\delta}^+} t^{1-2s} \psi^2 dt dx \end{aligned}$$

for some positive $C(c, \beta, \sigma, R_\delta) > 0$ depending only on c, σ, R_δ , and β . By Hölder inequality and Lemma 2.6, we have that

$$\begin{aligned} & \int_{B'_{R_\delta}} (|w|^{2^*(s)-2} + 1)(\eta\psi)^2 dx \\ & \leq \tilde{S}_{N,s} \left[\left(\int_{B'_{R_\delta}} |w|^{2^*(s)} dx \right)^{\frac{2^*(s)-2}{2^*(s)}} + |B'_{R_\delta}|^{\frac{2s}{N}} \right] \int_{B_{R_\delta}^+} t^{1-2s} |\nabla(\eta\psi)|^2 dt dx. \end{aligned}$$

We deduce that

$$(42) \quad A \int_{B_{R_\delta}^+} t^{1-2s} |\nabla(\eta\psi)|^2 dt dx - \kappa_s(\lambda + \delta) \int_{B'_{R_\delta}} |x|^{-2s} (\eta\psi)^2 dx \leq \text{const} \int_{B_{R_\delta}^+} t^{1-2s} \psi^2 dt dx,$$

for some positive $\text{const} > 0$ depending only on $C_f, s, p, N, \beta, \sigma, R_\delta$, where

$$A = \frac{1}{(1+\sigma)\beta} \left(1 - \frac{\sigma}{2} \right) - c\tilde{S}_{N,s} \left[\left(\int_{B'_{R_\delta}} |w|^{2^*(s)} dx \right)^{\frac{2^*(s)-2}{2^*(s)}} + |B'_{R_\delta}|^{\frac{2s}{N}} \right].$$

From Hardy inequality (17), we have that

$$\begin{aligned} A \int_{B_{R_\delta}^+} t^{1-2s} |\nabla(\eta\psi)|^2 dt dx - \kappa_s(\lambda + \delta) \int_{B'_{R_\delta}} |x|^{-2s} (\eta\psi)^2 dx \\ \geq \left(A - \frac{\lambda + \delta}{\Lambda_{N,s}} \right) \int_{B_{R_\delta}^+} t^{1-2s} |\nabla(\eta\psi)|^2 dt dx, \end{aligned}$$

and, by (40), we can choose β sufficiently close to 1 and σ, R_δ sufficiently small such that

$$A - \frac{\lambda + \delta}{\Lambda_{N,s}} > 0.$$

Hence we have that

$$C \int_{B_{R_\delta}^+} t^{1-2s} |\nabla(\eta\psi)|^2 dt dx \leq \int_{B_{R_\delta}^+} t^{1-2s} \psi^2 dt dx,$$

for some constant $C > 0$ depending on $f, h, s, p, N, \beta, \varepsilon, w, \lambda, \delta$. From Lemma 2.6 it follows that

$$C\tilde{S}_{N,s}^{-1} \left(\int_{B'_{R_\delta}} |\eta F_L(w)|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \leq \int_{B_{R_\delta}^+} t^{1-2s} |w|^{2\beta} dt dx \quad \text{for all } L \geq 0.$$

Hence by taking the limit as $L \rightarrow +\infty$

$$(43) \quad C\tilde{S}_{N,s}^{-1} \left(\int_{B'_{R_\delta/2}} |w|^{\beta 2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \leq \int_{B_{R_\delta}^+} t^{1-2s} |w|^{2\beta} dt dx.$$

The conclusion follows since $\beta > 1$ and $H^1(B_R^+; t^{1-2s}) \hookrightarrow L^q(B_R^+; t^{1-2s})$ for some $q > 2$, see for instance [[7], Theorem 1.2]. ■

Remark 3.5. From Lemma 3.4, we deduce that, if $w \in H^1(B_R^+; t^{1-2s})$ is a weak solution to (34), then $\frac{\lambda}{|x|^{2s}} + h + \frac{f(x,w)}{w} \in L_{\text{loc}}^q(B_R \setminus \{0\})$ for some $\frac{N}{2s} < q \leq \frac{p_0}{p-2}$ and hence, from Lemma 3.2, we conclude that $w \in C_{\text{loc}}^{0,\alpha}(\overline{B_r^+} \setminus \{0\})$, $\nabla_x v \in C_{\text{loc}}^{0,\beta}(\overline{B_r^+} \setminus \{0\})$, and $t^{1-2s} v_t \in C_{\text{loc}}^{0,\beta}(\overline{B_r^+} \setminus \{0\})$ for all $r \in (0, R)$ and some $0 < \beta < \alpha < 1$.

The following Pohozaev-type identity holds.

Theorem 3.6. *Let w solves (35). Then for a.e. $r \in (0, R)$ there holds*

$$\begin{aligned}
 (44) \quad & -\frac{N-2s}{2} \left[\int_{B_r^+} t^{1-2s} |\nabla w|^2 dz - \kappa_s \lambda \int_{B_r'} \frac{w^2}{|x|^{2s}} dx \right] \\
 & + \frac{r}{2} \left[\int_{S_r^+} t^{1-2s} |\nabla w|^2 dS - \kappa_s \lambda \int_{\partial B_r'} \frac{w^2}{|x|^{2s}} dS' \right] \\
 & = r \int_{S_r^+} t^{1-2s} \left| \frac{\partial w}{\partial \nu} \right|^2 dS - \frac{\kappa_s}{2} \int_{B_r'} (Nh + \nabla h \cdot x) w^2 dx + \frac{r\kappa_s}{2} \int_{\partial B_r'} h w^2 dS' \\
 & \quad + r\kappa_s \int_{\partial B_r'} F(x, w) dS' - \kappa_s \int_{B_r'} [\nabla_x F(x, w) \cdot x + NF(x, w)] dx
 \end{aligned}$$

and

$$\begin{aligned}
 (45) \quad & \int_{B_r^+} t^{1-2s} |\nabla w|^2 dz - \kappa_s \lambda \int_{B_r'} \frac{w^2}{|x|^{2s}} dx \\
 & = \int_{S_r^+} t^{1-2s} \frac{\partial w}{\partial \nu} w dS + \kappa_s \int_{B_r'} \left(h w^2 + f(x, w) w \right) dx.
 \end{aligned}$$

PROOF. We write our problem in the form

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla w) = 0, & \text{in } B_R^+, \\ -\lim_{t \rightarrow 0^+} t^{1-2s} w_t = G(x, w), & \text{on } B_R', \end{cases}$$

where $G \in C^1(B_R' \setminus \{0\} \times \mathbb{R})$, $G(x, \varrho) = \kappa_s \left(\frac{\lambda}{|x|^{2s}} \varrho + h(x) \varrho + f(x, \varrho) \right)$.

We have, on B_R^+ , the formula

$$(46) \quad \operatorname{div} \left(\frac{1}{2} t^{1-2s} |\nabla w|^2 z - t^{1-2s} (z \cdot \nabla w) \nabla w \right) = \frac{N-2s}{2} t^{1-2s} |\nabla w|^2 - (z \cdot \nabla w) \operatorname{div}(t^{1-2s} \nabla w).$$

Let $\rho < r < R$. Now we integrate by parts over the set $O_\delta := (B_r^+ \setminus \overline{B_\rho^+}) \cap \{(t, x), t > \delta\}$ with $\delta > 0$. We have

$$\begin{aligned}
 & \frac{N-2s}{2} \int_{O_\delta} t^{1-2s} |\nabla w(z)|^2 dz = -\frac{1}{2} \delta^{2-2s} \int_{B_r' \setminus \sqrt{r^2-\delta^2}}^{\sqrt{r^2-\delta^2}} |\nabla w|^2(\delta, x) dx \\
 & \quad + \delta^{2-2s} \int_{B_r' \setminus \sqrt{r^2-\delta^2}}^{\sqrt{r^2-\delta^2}} |w_t|^2(\delta, x) dx \\
 & \quad + \frac{r}{2} \int_{S_r^+ \cap \{t > \delta\}} t^{1-2s} |\nabla w|^2 dS - r \int_{S_r^+ \cap \{t > \delta\}} t^{1-2s} \left| \frac{\partial w}{\partial \nu} \right|^2 dS \\
 & \quad - \frac{\rho}{2} \int_{S_\rho^+ \cap \{t > \delta\}} t^{1-2s} |\nabla w|^2 dS + \rho \int_{S_\rho^+ \cap \{t > \delta\}} t^{1-2s} \left| \frac{\partial w}{\partial \nu} \right|^2 dS \\
 & \quad + \int_{B_r' \setminus \sqrt{r^2-\delta^2}}^{\sqrt{r^2-\delta^2}} (x \cdot \nabla_x w(\delta, x)) \delta^{1-2s} w_t(\delta, x) dx.
 \end{aligned}$$

We now claim that there exists a sequence $\delta_n \rightarrow 0$ such that

$$\lim_{n \rightarrow \infty} \left[\frac{1}{2} \delta_n^{2-2s} \int_{B_r'} |\nabla w|^2(\delta_n, x) dx + \delta_n^{2-2s} \int_{B_r'} |w_t|^2(\delta_n, x) dx \right] = 0.$$

If no such sequence exists, we would have

$$\liminf_{\delta \rightarrow 0} \left[\frac{1}{2} \delta^{2-2s} \int_{B_r'} |\nabla w|^2(\delta, x) dx + \delta^{2-2s} \int_{B_r'} |w_t|^2(\delta, x) dx \right] \geq C > 0$$

and thus there exists $\delta_0 > 0$ such that

$$\frac{1}{2} \delta^{2-2s} \int_{B_r'} |\nabla w|^2(\delta, x) dx + \delta^{2-2s} \int_{B_r'} |w_t|^2(\delta, x) dx \geq \frac{C}{2} \quad \text{for all } \delta \in (0, \delta_0).$$

It follows that

$$\frac{1}{2}\delta^{1-2s} \int_{B'_r} |\nabla w|^2(\delta, x) dx + \delta^{1-2s} \int_{B'_r} |w_t|^2(\delta, x) dx \geq \frac{C}{2\delta} \quad \text{for all } \delta \in (0, \delta_0)$$

and so integrating the above inequality on $(0, \delta_0)$ we contradict the fact that $w \in H^1(B_R^+; t^{1-2s})$.

Next, from the Dominated Convergence Theorem, Lemma 3.2 and Remark 3.5, we have that

$$\lim_{\delta \rightarrow 0} \int_{B'_r \setminus \sqrt{r^2 - \delta^2} \setminus B'_{\sqrt{\rho^2 - \delta^2}}} (x \cdot \nabla_x w(\delta, x)) \delta^{1-2s} w_t(\delta, x) dx = - \int_{B'_r \setminus B'_\rho} (x \cdot \nabla_x w) G(x, w) dx.$$

We conclude that (replacing O_δ with O_{δ_n} , for a sequence $\delta_n \rightarrow 0$) that

$$(47) \quad \begin{aligned} \frac{N-2s}{2} \int_{B_r^+ \setminus B_\rho^+} t^{1-2s} |\nabla w(z)|^2 dz &= \frac{r}{2} \int_{S_r^+} t^{1-2s} |\nabla w|^2 dS - r \int_{S_r^+} t^{1-2s} \left| \frac{\partial w}{\partial \nu} \right|^2 dS \\ &\quad - \frac{\rho}{2} \int_{S_\rho^+} t^{1-2s} |\nabla w|^2 dS - \rho \int_{S_\rho^+} t^{1-2s} \left| \frac{\partial w}{\partial \nu} \right|^2 dS - \int_{B'_r \setminus B'_\rho} (x \cdot \nabla_x w) G(x, w) dx. \end{aligned}$$

Furthermore, integration by parts yields

$$(48) \quad \begin{aligned} \int_{B'_r \setminus B'_\rho} (x \cdot \nabla_x w) G(x, w) dx &= -\frac{N-2s}{2} \kappa_s \lambda \int_{B'_r \setminus B'_\rho} \frac{w^2}{|x|^{2s}} dx \\ &\quad - \frac{\kappa_s}{2} \int_{B'_r \setminus B'_\rho} (Nh(x) + \nabla h(x) \cdot x) w^2 dx + \kappa_s \lambda \frac{r}{2} \int_{\partial B'_r} \frac{w^2}{|x|^{2s}} dS' \\ &\quad + \frac{r\kappa_s}{2} \int_{\partial B'_r} h(x) w^2 dS' - \kappa_s \lambda \frac{\rho}{2} \int_{\partial B'_\rho} \frac{w^2}{|x|^{2s}} dS' - \frac{\rho\kappa_s}{2} \int_{\partial B'_\rho} h(x) w^2 dS' \\ &\quad - \kappa_s \int_{B'_r \setminus B'_\rho} [\nabla_x F(x, w) \cdot x + NF(x, w)] dx \\ &\quad + r\kappa_s \int_{\partial B'_r} F(x, w) dS' - \rho\kappa_s \int_{\partial B'_\rho} F(x, w) dS'. \end{aligned}$$

Since $w \in H^1(B_R^+; t^{1-2s})$, in view of Lemma 2.5 and (8), there exists a sequence $\rho_n \rightarrow 0$ such that

$$\lim_{n \rightarrow \infty} \rho_n \left[\int_{S_{\rho_n}^+} t^{1-2s} |\nabla w|^2 dS + \int_{\partial B'_{\rho_n}} \frac{w^2}{|x|^{2s}} dS' + \int_{\partial B'_{\rho_n}} |F(x, w)| dS' \right] = 0.$$

Hence, taking $\rho = \rho_n$ and letting $n \rightarrow \infty$ in (47) and (48), we obtain (44).

(45) follows from Lemma 3.1 and density of $C^\infty(\overline{B_r^+})$ in $H^1(B_r^+; t^{1-2s})$. ■

For every $r \in (0, R]$ we define

$$(49) \quad D(r) = \frac{1}{r^{N-2s}} \left[\int_{B_r^+} t^{1-2s} |\nabla w|^2 dt dx - \kappa_s \int_{B'_r} \left(\frac{\lambda}{|x|^{2s}} w^2 + hw^2 + f(x, w) \right) dx \right]$$

and

$$(50) \quad H(r) = \frac{1}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} w^2 dS = \int_{\mathbb{S}_+^N} \theta_1^{1-2s} w^2(r\theta) dS.$$

Lemma 3.7. $H \in C^1(0, R)$ and

$$(51) \quad H'(r) = \frac{2}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} w \frac{\partial w}{\partial \nu} dS, \quad \text{for every } r \in (0, R),$$

$$(52) \quad H'(r) = \frac{2}{r} D(r), \quad \text{for every } r \in (0, R).$$

PROOF. Fix $r_0 \in (0, R)$ and consider the limit

$$(53) \quad \lim_{r \rightarrow r_0} \frac{H(r) - H(r_0)}{r - r_0} = \lim_{r \rightarrow r_0} \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \frac{|w(r\theta)|^2 - |w(r_0\theta)|^2}{r - r_0} dS.$$

Since $w \in C^1(\mathbb{R}_+^{N+1})$, then, for every $\theta \in \mathbb{S}_+^N$,

$$(54) \quad \lim_{r \rightarrow r_0} \frac{|w(r\theta)|^2 - |w(r_0\theta)|^2}{r - r_0} = 2 \frac{\partial w}{\partial \nu}(r_0\theta) w(r_0\theta).$$

On the other hand, for any $r \in (r_0/2, R)$ and $\theta \in \mathbb{S}_+^N$ we have

$$\left| \frac{|w(r\theta)|^2 - |w(r_0\theta)|^2}{r - r_0} \right| \leq 2 \sup_{B_R^+ \setminus B_{\frac{r_0}{2}}^+} |w| \cdot \sup_{B_R^+ \setminus B_{\frac{r_0}{2}}^+} \left| \frac{\partial w}{\partial \nu} \right| \leq 2 \sup_{B_R^+ \setminus B_{\frac{r_0}{2}}^+} |w| \cdot \sup_{B_R^+ \setminus B_{\frac{r_0}{2}}^+} \left(\left| \frac{t}{|z|} w_t \right| + \left| \frac{\nabla_x w \cdot x}{|z|} \right| \right)$$

and hence, by (53), (54), Lemma 3.2 and the Dominated Convergence Theorem, we obtain that

$$H'(r_0) = 2 \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \frac{\partial w}{\partial \nu}(r_0\theta) w(r_0\theta) dS(\theta) = \frac{2}{r_0^{N+1-2s}} \int_{S_{r_0}^+} t^{1-2s} w \frac{\partial w}{\partial \nu} dS.$$

The continuity of H' on the interval $(0, R)$ follows by the representation of H' given above, Lemma 3.2, and the Dominated Convergence Theorem.

Finally, (52) follows from (51), (49), and (45). ■

The regularity of the function D is established in the following lemma.

Lemma 3.8. *The function D defined in (49) belongs to $W_{\text{loc}}^{1,1}(0, R)$ and*

$$(55) \quad \begin{aligned} D'(r) = & \frac{2}{r^{N+1-2s}} \left[r \int_{S_r^+} t^{1-2s} \left| \frac{\partial w}{\partial \nu} \right|^2 dS - \kappa_s \int_{B_r'} \left(sh + \frac{1}{2} (\nabla h \cdot x) \right) w^2 dx \right] \\ & + \frac{\kappa_s}{r^{N+1-2s}} \int_{B_r'} ((N-2s)f(x, w)w - 2NF(x, w) - 2\nabla_x F(x, w) \cdot x) dx \\ & + \frac{\kappa_s}{r^{N-2s}} \int_{\partial B_r'} (2F(x, w) - f(x, w)w) dS' \end{aligned}$$

in a distributional sense and for a.e. $r \in (0, R)$.

PROOF. For any $r \in (0, r_0)$ let

$$(56) \quad I(r) = \int_{B_r^+} t^{1-2s} |\nabla w|^2 dt dx - \kappa_s \int_{B_r'} \left(\frac{\lambda}{|x|^{2s}} w^2 + hw^2 + f(x, w)w \right) dx.$$

From the fact that $w \in H^1(B_R^+; t^{1-2s})$, Lemma 2.5, and (8), we deduce that $I \in W^{1,1}(0, R)$ and

$$(57) \quad I'(r) = \int_{S_r^+} t^{1-2s} |\nabla w|^2 dS - \kappa_s \int_{\partial B_r'} \left(\frac{\lambda}{|x|^{2s}} w^2 + hw^2 + f(x, w)w \right) dS'$$

for a.e. $r \in (0, R)$ and in the distributional sense. Therefore $D \in W_{\text{loc}}^{1,1}(0, R)$ and, using (44), (56), and (57) into

$$D'(r) = r^{2s-1-N} [-(N-2s)I(r) + rI'(r)],$$

we obtain (55) for a.e. $r \in (0, R)$ and in the distributional sense. ■

Before going on, we recall that w is nontrivial and satisfies (35). We prove now that, if $w \not\equiv 0$, $H(r)$ does not vanish for r sufficiently small.

Lemma 3.9. *There exists $R_0 \in (0, R)$ such that $H(r) > 0$ for any $r \in (0, R_0)$, where H is defined by (50).*

PROOF. Clearly from assumption (2), there exists $R_0 \in (0, R)$ such that

$$(58) \quad \frac{\lambda}{\Lambda_{N,s}} + \frac{C_h R_0^\varepsilon}{\Lambda_{N,s}} + C_f S_{N,s}^{-1} \left(\frac{\omega_{N-1}}{N} \right)^{\frac{2^*(s)-p}{2^*(s)}} R_0^{\frac{N(2^*(s)-p)}{2^*(s)}} \|w\|_{L^{2^*(s)}(B_{R_0}')}^{p-2} < 1,$$

where ω_{N-1} denotes the volume of the unit sphere \mathbb{S}^{N-1} , i.e. $\omega_{N-1} = \int_{\mathbb{S}^{N-1}} dS$.

Next suppose by contradiction that there exists $r_0 \in (0, R_0)$ such that $H(r_0) = 0$. Then $w = 0$ a.e. on $S_{r_0}^+$. From (45) it follows that

$$\int_{B_{r_0}^+} t^{1-2s} |\nabla w|^2 dz - \kappa_s \lambda \int_{B_{r_0}'} \frac{w^2}{|x|^{2s}} dx - \kappa_s \int_{B_{r_0}'} \left(h(x)w^2 + f(x, w)w \right) dx = 0.$$

From Lemma 2.5, assumptions (3)-(4), Hölder's inequality, and (8), it follows that

$$\begin{aligned} 0 &= \int_{B_{r_0}^+} t^{1-2s} |\nabla w|^2 dz - \kappa_s \lambda \int_{B_{r_0}'} \frac{w^2}{|x|^{2s}} dx - \kappa_s \int_{B_{r_0}'} \left(h(x)w^2 + f(x, w)w \right) dx \\ &\geq \left[1 - \frac{\lambda}{\Lambda_{N,s}} - \frac{C_h R_0^\varepsilon}{\Lambda_{N,s}} - C_f S_{N,s}^{-1} \left(\frac{\omega_{N-1}}{N} \right)^{\frac{2^*(s)-p}{2^*(s)}} R_0^{\frac{N(2^*(s)-p)}{2^*(s)}} \|w\|_{L^{2^*(s)}(B_{R_0}') }^{p-2} \right] \int_{B_{r_0}^+} t^{1-2s} |\nabla w|^2 dz, \end{aligned}$$

which, together with (58), implies $w \equiv 0$ in $B_{r_0}^+$ by Lemma 2.4. Classical unique continuation principles for second order elliptic equations with locally bounded coefficients (see e.g. [24]) allow to conclude that $w = 0$ a.e. in B_R^+ , a contradiction. ■

Letting R_0 be as in Lemma 3.9 and recalling (36), the *Almgren type frequency function*

$$(59) \quad \mathcal{N}(r) = \frac{D(r)}{H(r)}$$

is well defined in $(0, R_0)$. Using Lemmas 3.7 and 3.8, we can now compute the derivative of \mathcal{N} .

Lemma 3.10. *The function \mathcal{N} defined in (59) belongs to $W_{\text{loc}}^{1,1}(0, R_0)$ and*

$$(60) \quad \mathcal{N}'(r) = \nu_1(r) + \nu_2(r)$$

in a distributional sense and for a.e. $r \in (0, R_0)$, where

$$(61) \quad \nu_1(r) = \frac{2r \left[\left(\int_{S_r^+} t^{1-2s} \left| \frac{\partial w}{\partial \nu} \right|^2 dS \right) \cdot \left(\int_{S_r^+} t^{1-2s} w^2 dS \right) - \left(\int_{S_r^+} t^{1-2s} w \frac{\partial w}{\partial \nu} dS \right)^2 \right]}{\left(\int_{S_r^+} t^{1-2s} w^2 dS \right)^2}$$

and

$$(62) \quad \begin{aligned} \nu_2(r) &= -\kappa_s \frac{\int_{B_r'} (2sh + \nabla h \cdot x) |w|^2 dx}{\int_{S_r^+} t^{1-2s} w^2 dS} + \kappa_s \frac{r \int_{\partial B_r'} (2F(x, w) - f(x, w)w) dS'}{\int_{S_r^+} t^{1-2s} w^2 dS} \\ &\quad + \kappa_s \frac{\int_{B_r'} ((N-2s)f(x, w)w - 2NF(x, w) - 2\nabla_x F(x, w) \cdot x) dx}{\int_{S_r^+} t^{1-2s} w^2 dS}. \end{aligned}$$

PROOF. From Lemmas 3.7, 3.9, and 3.8, it follows that $\mathcal{N} \in W_{\text{loc}}^{1,1}(0, R_0)$. From (52) it follows that

$$\mathcal{N}'(r) = \frac{D'(r)H(r) - D(r)H'(r)}{(H(r))^2} = \frac{D'(r)H(r) - \frac{1}{2}r(H'(r))^2}{(H(r))^2}$$

and the proof of the lemma easily follows from (51) and (55). ■

We now prove that $\mathcal{N}(r)$ admits a finite limit as $r \rightarrow 0^+$. To this aim, the following estimate plays a crucial role.

Lemma 3.11. *Let \mathcal{N} be the function defined in (59). There exist $\tilde{R} \in (0, R_0)$ and a constant $\overline{C} > 0$ such that*

$$(63) \quad \begin{aligned} \int_{B_r^+} t^{1-2s} |\nabla w|^2 dt dx - \kappa_s \int_{B_r'} \left(\frac{\lambda}{|x|^{2s}} w^2 + hw^2 + f(x, w)w \right) dx \\ \geq - \left(\frac{N-2s}{2r} \right) \int_{S_r^+} t^{1-2s} w^2 dS + \overline{C} \left(\int_{B_r'} \frac{w^2}{|x|^{2s}} dx + \left(\int_{B_r'} |w|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \right), \end{aligned}$$

$$(64) \quad \begin{aligned} \int_{B_r^+} t^{1-2s} |\nabla w|^2 dt dx - \kappa_s \int_{B_r'} \left(\frac{\lambda}{|x|^{2s}} w^2 + hw^2 + f(x, w)w \right) dx \\ \geq - \left(\frac{N-2s}{2r} \right) \int_{S_r^+} t^{1-2s} w^2 dS + \overline{C} \int_{B_r^+} t^{1-2s} |\nabla w|^2 dt dx, \end{aligned}$$

and

$$(65) \quad \mathcal{N}(r) > -\frac{N-2s}{2}$$

for every $r \in (0, \tilde{R})$.

PROOF. From Corollary 2.7, (3), and (4), it follows that

$$\begin{aligned} & \int_{B_r^+} t^{1-2s} |\nabla w|^2 dt dx - \kappa_s \int_{B_r'} \left(\frac{\lambda}{|x|^{2s}} w^2 + h w^2 + f(x, w) w \right) dx + \left(\frac{N-2s}{2r} \right) \int_{S_r^+} t^{1-2s} w^2 dS \\ & \geq \left(\frac{\kappa_s (\Lambda_{N,s} - \lambda)}{2} - C_h \kappa_s r^\varepsilon \right) \int_{B_r'} \frac{w^2}{|x|^{2s}} dx \\ & \quad + \left(\frac{\Lambda_{N,s} - \lambda}{2(1 + \Lambda_{N,s}) \tilde{S}_{N,s}} - C_f \kappa_s |B_r'|^{\frac{2^*(s)-p}{2^*(s)}} \|w\|_{L^{2^*(s)}(B_r')}^{p-2} \right) \left(\int_{B_r'} |w|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \end{aligned}$$

for every $r \in (0, R_0)$. Since $\lambda < \Lambda_{N,s}$, from the above estimate it follows that we can choose $\tilde{R} \in (0, R_0)$ sufficiently small such that estimate (63) holds for $r \in (0, \tilde{R})$ for some positive constant $\bar{C} > 0$. The proof of (64) can be performed in a similar way, using Lemmas 2.5 and 2.6. Estimate (63), together with (49) and (50), yields (65). ■

Lemma 3.12. *Let \tilde{R} be as in Lemma 3.11 and ν_2 as in (62). Then there exist a positive constant $C_1 > 0$ and a function $g \in L^1(0, \tilde{R})$, $g \geq 0$ a.e. in $(0, \tilde{R})$, such that*

$$|\nu_2(r)| \leq C_1 \left[\mathcal{N}(r) + \frac{N-2s}{2} \right] \left(r^{-1+\varepsilon} + r^{-1+\frac{2s(p_0-2^*(s))}{p_0}} + g(r) \right)$$

for a.e. $r \in (0, \tilde{R})$ and

$$\int_0^r g(\rho) d\rho \leq \frac{1}{1-\alpha} \|w\|_{L^p(B_{\tilde{R}}')}^{p(1-\alpha)} r^{N \left(\frac{\alpha 2^*(s)-2}{2^*(s)} - \frac{p\alpha-2}{p_0} \right)}$$

for all $r \in (0, \tilde{R})$ with some α satisfying $\frac{2}{p} < \alpha < 1$.

PROOF. From (3) and (63) we deduce that

$$\begin{aligned} \left| \int_{B_r'} (2sh(x) + \nabla h(x) \cdot x) |w|^2 dx \right| & \leq 2C_h r^\varepsilon \int_{B_r'} \frac{|w|^2}{|x|^{2s}} dx \\ & \leq 2C_h \bar{C}^{-1} r^{\varepsilon+N-2s} [D(r) + \frac{N-2s}{2} H(r)], \end{aligned}$$

and, therefore, for any $r \in (0, \tilde{R})$, we have that

$$\begin{aligned} (66) \quad \left| \frac{\int_{B_r'} (2sh(x) + \nabla h(x) \cdot x) |w|^2 dx}{\int_{S_r^+} t^{1-2s} w^2 dS} \right| & \leq 2C_h \bar{C}^{-1} r^{-1+\varepsilon} \frac{D(r) + \frac{N-2s}{2} H(r)}{H(r)} \\ & = 2C_h \bar{C}^{-1} r^{-1+\varepsilon} \left[\mathcal{N}(r) + \frac{N-2s}{2} \right]. \end{aligned}$$

By (4), Hölder's inequality, and (63), for some constant $\text{const} = \text{const}(N, s, C_f) > 0$ depending on N, s, C_f , and for all $r \in (0, \tilde{R})$, there holds

$$\begin{aligned} & \left| \int_{B_r'} ((N-2s)f(x, w)w - 2NF(x, w) - 2\nabla_x F(x, w) \cdot x) dx \right| \\ & \leq \text{const} \int_{B_r'} (|w|^2 + |w|^{2^*(s)}) dx \\ & \leq \text{const} \left(\left(\frac{\omega_{N-1}}{N} \right)^{\frac{2s}{N}} r^{2s} + \|w\|_{L^{2^*(s)}(B_{\tilde{R}}')}^{2^*(s)-2} \right) \left(\int_{B_r'} |w|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \\ & \leq \frac{\text{const}}{\bar{C}} \left(\left(\frac{\omega_{N-1}}{N} \right)^{\frac{2s}{N}} r^{2s} + \left(\frac{\omega_{N-1}}{N} \right)^{\frac{2s(p_0-2^*(s))}{Np_0}} r^{\frac{2s(p_0-2^*(s))}{p_0}} \|w\|_{L^{p_0}(B_{\tilde{R}}')}^{2^*(s)-2} \right) r^{N-2s} [D(r) + \frac{N-2s}{2} H(r)] \end{aligned}$$

and hence

$$(67) \quad \left| \frac{\int_{B'_r} ((N-2s)f(x,w)w - 2NF(x,w) - 2\nabla_x F(x,w) \cdot x) dx}{\int_{S_r^+} t^{1-2s} w^2 dS} \right| \\ \leq \frac{\text{const}}{\overline{C}} \left(\left(\frac{\omega_{N-1}}{N} \right)^{\frac{2s}{N}} r^{\frac{2s2^*(s)}{p_0}} + \left(\frac{\omega_{N-1}}{N} \right)^{\frac{2s(p_0-2^*(s))}{Np_0}} \|w\|_{L^{p_0}(B'_R)}^{2^*(s)-2} \right) r^{-1+\frac{2s(p_0-2^*(s))}{p_0}} [\mathcal{N}(r) + \frac{N-2s}{2}].$$

Let us fix $\frac{2}{p} < \alpha < 1$. By Hölder's inequality and (63),

$$(68) \quad \left(\int_{B'_r} |w|^p dx \right)^\alpha = \left(\int_{B'_r} |w|^{p-\frac{2}{\alpha}} |w|^{\frac{2}{\alpha}} dx \right)^\alpha \\ \leq \left(\int_{B'_r} |w|^{2^*(s)\frac{p\alpha-2}{2^*(s)\alpha-2}} dx \right)^{\frac{\alpha 2^*(s)-2}{2^*(s)}} \left(\int_{B'_r} |w|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \\ \leq \left(\frac{\omega_{N-1}}{N} \right)^{\frac{\alpha 2^*(s)-2}{2^*(s)} - \frac{p\alpha-2}{p_0}} r^{N\left(\frac{\alpha 2^*(s)-2}{2^*(s)} - \frac{p\alpha-2}{p_0}\right)} \|w\|_{L^{p_0}(B'_R)}^{p\alpha-2} \frac{r^{N-2s}}{\overline{C}} [D(r) + \frac{N-2s}{2}H(r)] \\ = \overline{C}^{-1} \left(\frac{\omega_{N-1}}{N} \right)^{\frac{\beta}{N}} r^{-1+\beta} \left[\mathcal{N}(r) + \frac{N-2s}{2} \right] \left(\int_{S_r^+} t^{1-2s} w^2 dS \right)$$

for all $r \in (0, \tilde{R})$, where $\beta = N\left(\frac{\alpha 2^*(s)-2}{2^*(s)} - \frac{p\alpha-2}{p_0}\right) > 0$. From (4), (68), and (65), there exists some $\text{const} = \text{const}(N, s, C_f) > 0$ depending on N, s, C_f such that, for all $r \in (0, \tilde{R})$,

$$(69) \quad \left| \frac{r \int_{\partial B'_r} (2F(x,w) - f(x,w)w) dS'}{\int_{S_r^+} t^{1-2s} w^2 dS} \right| \leq \text{const} \frac{r \int_{\partial B'_r} |w|^p dS'}{\int_{S_r^+} t^{1-2s} w^2 dS} \\ \leq \frac{\text{const}}{\overline{C}} \left(\frac{\omega_{N-1}}{N} \right)^{\frac{\beta}{N}} \left[\mathcal{N}(r) + \frac{N-2s}{2} \right] \frac{r^\beta \int_{\partial B'_r} |w|^p dS'}{\left(\int_{B'_r} |w|^p dx \right)^\alpha}.$$

By a direct calculation, we have that

$$(70) \quad \frac{r^\beta \int_{\partial B'_r} |w|^p dS'}{\left(\int_{B'_r} |w|^p dx \right)^\alpha} = \frac{1}{1-\alpha} \left[\frac{d}{dr} \left(r^\beta \left(\int_{B'_r} |w|^p dx \right)^{1-\alpha} \right) - \beta r^{-1+\beta} \left(\int_{B'_r} |w|^p dx \right)^{1-\alpha} \right]$$

in the distributional sense and for a.e. $r \in (0, \tilde{R})$. Since

$$\lim_{r \rightarrow 0^+} r^\beta \left(\int_{B'_r} |w|^p dx \right)^{1-\alpha} = 0$$

we deduce that the function

$$r \mapsto \frac{d}{dr} \left(r^\beta \left(\int_{B'_r} |w|^p dx \right)^{1-\alpha} \right)$$

is integrable over $(0, \tilde{R})$. Being

$$r^{-1+\beta} \left(\int_{B'_r} |w|^p dx \right)^{1-\alpha} = o(r^{-1+\beta})$$

as $r \rightarrow 0^+$, we have that also the function

$$r \mapsto r^{-1+\beta} \left(\int_{B'_r} |w|^p dx \right)^{\frac{p-2}{p}}$$

is integrable over $(0, \tilde{R})$. Therefore, by (70), we deduce that

$$(71) \quad g(r) := \frac{r^\beta \int_{\partial B'_r} |w|^p dS'}{\left(\int_{B'_r} |w|^p dx \right)^\alpha} \in L^1(0, \tilde{R})$$

and

$$(72) \quad 0 \leq \int_0^r g(\rho) d\rho \leq \frac{1}{1-\alpha} \|w\|_{L^p(B'_r)}^{p(1-\alpha)} r^\beta$$

for all $r \in (0, \tilde{R})$. Collecting (66), (67), (69), (71), and (72), we obtain the stated estimate. ■

Lemma 3.13. *Let \tilde{R} be as in Lemma 3.11 and \mathcal{N} as in (59). Then there exist a positive constant $C_2 > 0$ such that*

$$(73) \quad \mathcal{N}(r) \leq C_2$$

for all $r \in (0, \tilde{R})$.

PROOF. By Lemma 3.10, Schwarz's inequality, and Lemma 3.12, we obtain

$$(74) \quad \left(\mathcal{N} + \frac{N-2s}{2} \right)'(r) \geq \nu_2(r) \geq -C_1 \left[\mathcal{N}(r) + \frac{N-2s}{2} \right] \left(r^{-1+\varepsilon} + r^{-1+\frac{2s(p_0-2^*(s))}{p_0}} + g(r) \right)$$

for a.e. $r \in (0, \tilde{R})$. Integration over (r, \tilde{R}) yields

$$\mathcal{N}(r) \leq -\frac{N-2s}{2} + \left(\mathcal{N}(\tilde{R}) + \frac{N-2s}{2} \right) \exp \left(C_1 \left(\frac{\tilde{R}^\varepsilon}{\varepsilon} + \frac{p_0 \tilde{R}^{\frac{2s(p_0-2^*(s))}{p_0}}}{2s(p_0-2^*(s))} + \int_0^{\tilde{R}} g(\rho) d\rho \right) \right)$$

for any $r \in (0, \tilde{R})$, thus proving estimate (73). ■

Lemma 3.14. *The limit*

$$\gamma := \lim_{r \rightarrow 0^+} \mathcal{N}(r)$$

exists and is finite.

PROOF. By Lemmas 3.12 and 3.13, the function ν_2 defined in (62) belongs to $L^1(0, \tilde{R})$. Hence, by Lemma 3.10 and Schwarz's inequality, \mathcal{N}' is the sum of a nonnegative function and of a L^1 -function on $(0, \tilde{R})$. Therefore

$$\mathcal{N}(r) = \mathcal{N}(\tilde{R}) - \int_r^{\tilde{R}} \mathcal{N}'(\rho) d\rho$$

admits a limit as $r \rightarrow 0^+$ which is necessarily finite in view of (65) and (73). ■

The function H defined in (50) can be estimated as follows.

Lemma 3.15. *Let $\gamma := \lim_{r \rightarrow 0^+} \mathcal{N}(r)$ be as in Lemma 3.14 and \tilde{R} as in Lemma 3.11. Then there exists a constant $K_1 > 0$ such that*

$$(75) \quad H(r) \leq K_1 r^{2\gamma} \quad \text{for all } r \in (0, \tilde{R}).$$

Moreover, for any $\sigma > 0$ there exists a constant $K_2(\sigma) > 0$ depending on σ such that

$$(76) \quad H(r) \geq K_2(\sigma) r^{2\gamma+\sigma} \quad \text{for all } r \in (0, \tilde{R}).$$

PROOF. By Lemma 3.14, $\mathcal{N}' \in L^1(0, \tilde{r})$ and, by Lemma 3.13, \mathcal{N} is bounded, then from (74) and (72) it follows that

$$(77) \quad \mathcal{N}(r) - \gamma = \int_0^r \mathcal{N}'(\rho) d\rho \geq -C_3 r^\delta$$

for some constant $C_3 > 0$ and all $r \in (0, \tilde{R})$, where

$$(78) \quad \delta = \min \left\{ \varepsilon, \frac{2s(p_0-2^*(s))}{p_0}, N \left(\frac{\alpha 2^*(s)-2}{2^*(s)} - \frac{p\alpha-2}{p_0} \right) \right\} > 0.$$

Therefore by (52), (59), and (77) we deduce that, for all $r \in (0, \tilde{R})$,

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} \geq \frac{2\gamma}{r} - 2C_3 r^{-1+\delta},$$

which, after integration over the interval (r, \tilde{R}) , yields (75).

Since $\gamma = \lim_{r \rightarrow 0^+} \mathcal{N}(r)$, for any $\sigma > 0$ there exists $r_\sigma > 0$ such that $\mathcal{N}(r) < \gamma + \sigma/2$ for any $r \in (0, r_\sigma)$ and hence

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} < \frac{2\gamma + \sigma}{r} \quad \text{for all } r \in (0, r_\sigma).$$

Integrating over the interval (r, r_σ) and by continuity of H outside 0, we obtain (76) for some constant $K_2(\sigma)$ depending on σ . ■

4. THE BLOW-UP ARGUMENT

The main result of this section, which also contains Theorem 1.1, is the following theorem.

Theorem 4.1. *Let w satisfy (35), with s, λ, h, f as in assumptions (2), (3), and (4). Then, letting $\mathcal{N}(r)$ as in (36), there there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that*

$$(79) \quad \lim_{r \rightarrow 0^+} \mathcal{N}(r) = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_{k_0}(\lambda)}.$$

Furthermore, if γ denotes the limit in (79), $m \geq 1$ is the multiplicity of the eigenvalue $\mu_{j_0}(\lambda) = \mu_{j_0+1}(\lambda) = \dots = \mu_{j_0+m-1}(\lambda)$ and $\{\psi_i\}_{i=j_0}^{j_0+m-1}$ ($j_0 \leq k_0 \leq j_0 + m - 1$) is an $L^2(\mathbb{S}_+^N; \theta_1^{1-2s})$ -orthonormal basis for the eigenspace of problem (10) associated to $\mu_{k_0}(\lambda)$, then

$$\begin{aligned} \tau^{-\gamma} w(\tau\theta) &\rightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{0,\alpha}(\mathbb{S}_+^N) \quad \text{as } \tau \rightarrow 0^+, \\ \tau^{-\gamma} w(0, \tau\theta') &\rightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(0, \theta') \quad \text{in } C^{1,\alpha}(\mathbb{S}^{N-1}) \quad \text{as } \tau \rightarrow 0^+, \end{aligned}$$

and

$$\tau^{1-\gamma} \nabla_x w(0, \tau\theta') \rightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \left(\gamma \psi_i(0, \theta') \theta' + \nabla_{\mathbb{S}^{N-1}} \psi_i(0, \cdot)(\theta') \right) \quad \text{in } C^{0,\alpha}(\mathbb{S}^{N-1}) \quad \text{as } \tau \rightarrow 0^+,$$

for some $\alpha \in (0, 1)$, where

$$\begin{aligned} \beta_i &= R^{-\gamma} \int_{\mathbb{S}_+^N} \theta_1^{1-2s} w(R\theta) \psi_i(\theta) dS(\theta) \\ &\quad + \kappa_s \int_{\mathbb{S}^{N-1}} \left[\int_0^R \frac{h(t\theta') w(0, t\theta') + f(t\theta', w(0, t\theta'))}{2\gamma + N - 2s} \left(t^{2s-\gamma-1} - \frac{t^{\gamma+N-1}}{R^{2\gamma+N-2s}} \right) dt \right] \psi_i(0, \theta') dS(\theta'), \end{aligned}$$

for all $R > 0$ such that $\overline{B'_R} = \{x \in \mathbb{R}^N : |x| \leq R\} \subset \Omega$ and $(\beta_{j_0}, \beta_{j_0+1}, \dots, \beta_{j_0+m-1}) \neq (0, 0, \dots, 0)$.

To prove Theorem 4.1, we start by determining the asymptotic profile of blowing up renormalized solutions to (34).

Lemma 4.2. *Let w as in Theorem 4.1. Let $\gamma := \lim_{r \rightarrow 0^+} \mathcal{N}(r)$ as in Lemma 3.14. Then*

- (i) *there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that $\gamma = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_{k_0}(\lambda)}$;*
- (ii) *for every sequence $\tau_n \rightarrow 0^+$, there exist a subsequence $\{\tau_{n_k}\}_{k \in \mathbb{N}}$ and an eigenfunction ψ of problem (10) associated to the eigenvalue $\mu_{k_0}(\lambda)$ such that $\|\psi\|_{L^2(\mathbb{S}_+^N; \theta_1^{1-2s})} = 1$ and*

$$\frac{w(\tau_{n_k} z)}{\sqrt{H(\tau_{n_k})}} \rightarrow |z|^\gamma \psi\left(\frac{z}{|z|}\right)$$

strongly in $H^1(B_r^+; t^{1-2s})$ and in $C_{\text{loc}}^{0,\alpha}(\overline{B_r^+} \setminus \{0\})$ for some $\alpha \in (0, 1)$ and all $r \in (0, 1)$ and

$$\frac{w(0, \tau_{n_k} x)}{\sqrt{H(\tau_{n_k})}} \rightarrow |x|^\gamma \psi\left(0, \frac{x}{|x|}\right)$$

in $C_{\text{loc}}^{1,\alpha}(B_1' \setminus \{0\})$.

PROOF. Let us set

$$(80) \quad w^\tau(z) = \frac{w(\tau z)}{\sqrt{H(\tau)}}.$$

We notice that $\int_{S_1^+} t^{1-2s} |w^\tau|^2 dS = 1$. Moreover, by scaling and (73),

$$(81) \quad \int_{B_1^+} t^{1-2s} |\nabla w^\tau(z)|^2 dz - \kappa_s \int_{B_1'} \left(\frac{\lambda}{|x|^{2s}} |w^\tau|^2 + \tau^{2s} h(\tau x) |w^\tau|^2 + \frac{\tau^{2s}}{\sqrt{H(\tau)}} f(\tau x, \sqrt{H(\tau)} w^\tau) w^\tau \right) dx = \mathcal{N}(\tau) \leq C_2$$

for every $\tau \in (0, \tilde{R})$, whereas, from (64),

$$(82) \quad \mathcal{N}(\tau) \geq \frac{\tau^{-N+2s}}{H(\tau)} \left(- \left(\frac{N-2s}{2\tau} \right) \int_{S_\tau^+} t^{1-2s} w^2 dS + \overline{C} \int_{B_\tau^+} t^{1-2s} |\nabla w|^2 dt dx \right) \\ = - \frac{N-2s}{2} + \overline{C} \int_{B_1^+} t^{1-2s} |\nabla w^\tau(z)|^2 dz$$

for every $\tau \in (0, \tilde{R})$. From (81) and (82) we deduce that

$$(83) \quad \{w^\tau\}_{\tau \in (0, \tilde{R})} \text{ is bounded in } H^1(B_1^+; t^{1-2s}).$$

Therefore, for any given sequence $\tau_n \rightarrow 0^+$, there exists a subsequence $\tau_{n_k} \rightarrow 0^+$ such that $w^{\tau_{n_k}} \rightharpoonup \tilde{w}$ weakly in $H^1(B_1^+; t^{1-2s})$ for some $\tilde{w} \in H^1(B_1^+; t^{1-2s})$. Due to compactness of the trace embedding (25), we obtain that $\int_{S_1^+} t^{1-2s} |\tilde{w}|^2 dS = 1$. In particular $\tilde{w} \not\equiv 0$.

For every small $\tau \in (0, \tilde{R})$, w^τ satisfies

$$(84) \quad \begin{cases} \operatorname{div}(t^{1-2s} \nabla w^\tau) = 0, & \text{in } B_1^+, \\ -\lim_{t \rightarrow 0^+} t^{1-2s} \frac{\partial w^\tau}{\partial t} = \kappa_s \left(\frac{\lambda}{|x|^{2s}} w^\tau + \tau^{2s} h(\tau x) w^\tau + \frac{\tau^{2s}}{\sqrt{H(\tau)}} f(\tau x, \sqrt{H(\tau)} w^\tau) \right), & \text{on } B_1', \end{cases}$$

in a weak sense, i.e.

$$(85) \quad \int_{B_1^+} t^{1-2s} \nabla w^\tau \cdot \nabla \tilde{\varphi} dt dx \\ = \kappa_s \int_{B_1'} \left(\frac{\lambda}{|x|^{2s}} w^\tau + \tau^{2s} h(\tau x) w^\tau + \frac{\tau^{2s}}{\sqrt{H(\tau)}} f(\tau x, \sqrt{H(\tau)} w^\tau) \right) \tilde{\varphi}(0, x) dx$$

for all $\tilde{\varphi} \in H^1(B_1^+; t^{1-2s})$ s.t. $\tilde{\varphi} = 0$ on \mathbb{S}_+^N and, for such $\tilde{\varphi}$, by (4) and Hölder's inequality,

$$(86) \quad \frac{\tau^{2s}}{\sqrt{H(\tau)}} \left| \int_{B_1'} f(\tau x, \sqrt{H(\tau)} w^\tau(0, x)) \tilde{\varphi}(0, x) dx \right| \\ \leq C_f \tau^{2s} \int_{B_1'} |w(0, \tau x)|^{p-2} |w^\tau(0, x)| |\tilde{\varphi}(0, x)| dx \\ \leq C_f \|\operatorname{Tr} \tilde{\varphi}\|_{L^p(B_1')} \|w^\tau\|_{L^p(B_1')} \|w\|_{L^p(B_1')}^{p-2} \tau^{\frac{(N-2s)(2^*(s)-p)}{p}} = o(1) \quad \text{as } \tau \rightarrow 0^+$$

and, by (3) and Lemma 2.5,

$$(87) \quad \tau^{2s} \left| \int_{B_1'} h(\tau x) w^\tau \tilde{\varphi}(0, x) dx \right| \\ \leq \frac{C_h \tau^\varepsilon}{\kappa_s \Lambda_{N,s}} \left(\int_{B_1^+} t^{1-2s} |\nabla w^\tau(z)|^2 dz + \frac{N-2s}{2} \right)^{1/2} \left(\int_{B_1^+} t^{1-2s} |\nabla \tilde{\varphi}(z)|^2 dz \right)^{1/2} = o(1) \quad \text{as } \tau \rightarrow 0^+.$$

From (86), (87), and weak convergence $w^{\tau_{n_k}} \rightharpoonup \tilde{w}$ in $H^1(B_1^+; t^{1-2s})$, we can pass to the limit in (84) along the sequence τ_{n_k} and obtain that \tilde{w} weakly solves

$$(88) \quad \begin{cases} \operatorname{div}(t^{1-2s} \nabla \tilde{w}) = 0, & \text{in } B_1^+, \\ -\lim_{t \rightarrow 0^+} t^{1-2s} \frac{\partial \tilde{w}}{\partial t} = \kappa_s \frac{\lambda}{|x|^{2s}} \tilde{w}, & \text{on } B_1'. \end{cases}$$

From (4), letting $q = \frac{p_0}{p-2} > \frac{N}{2s}$ with p_0 as in Lemma 3.4, we have that

$$\left\| \frac{\tau^{2s}}{\sqrt{H(\tau)}} \frac{f(\tau x, \sqrt{H(\tau)} w^\tau(x))}{w^\tau(x)} \right\|_{L^q(B'_2)} \leq C_f \tau^{2s - \frac{N}{q}} \left(\int_{B_{2\tau}} |w(x)|^{p_0} dx \right)^{1/q} = O(1)$$

as $\tau \rightarrow 0^+$. Therefore from Lemma 3.2 part (i) there holds

$$w^{\tau_{n_k}} \rightarrow \tilde{w} \quad \text{in } C_{\text{loc}}^{0,\alpha}(\overline{B_r^+} \setminus \{0\}),$$

while Lemma 3.2 part (ii) and Remark 3.3 imply that

$$(89) \quad \nabla_x w^{\tau_{n_k}} \rightarrow \nabla_x \tilde{w}, \quad \text{and} \quad t^{1-2s} \frac{\partial w^{\tau_{n_k}}}{\partial t} \rightarrow t^{1-2s} \frac{\partial \tilde{w}}{\partial t} \quad \text{in } C_{\text{loc}}^{0,\alpha}(\overline{B_r^+} \setminus \{0\})$$

for some $\alpha \in (0, 1)$ and all $r \in (0, 1)$. Reasoning as in (86), (87), we can prove that

$$(90) \quad \frac{\tau^{2s}}{\sqrt{H(\tau)}} \int_{B'_1} f(\tau x, \sqrt{H(\tau)} w^\tau) w^\tau dx = o(1) \quad \text{as } \tau \rightarrow 0^+$$

and

$$(91) \quad \tau^{2s} \int_{B'_1} h(\tau x) |w^\tau|^2 dx = o(1) \quad \text{as } \tau \rightarrow 0^+.$$

Multiplying equation (84) with w^τ , integrating in B_r^+ , and using (89), (90), (91), we easily obtain that $\|w^{\tau_{n_k}}\|_{H^1(B_r^+; t^{1-2s})} \rightarrow \|\tilde{w}\|_{H^1(B_r^+; t^{1-2s})}$ for all $r \in (0, 1)$, and hence

$$(92) \quad w^{\tau_{n_k}} \rightarrow \tilde{w} \quad \text{in } H^1(B_r^+; t^{1-2s})$$

for any $r \in (0, 1)$.

For any $r \in (0, 1)$ and $k \in \mathbb{N}$, let us define the functions

$$D_k(r) = \frac{1}{r^{N-2s}} \left[\int_{B_r^+} t^{1-2s} |\nabla w^{\tau_{n_k}}|^2 dt dx - \kappa_s \int_{B_r^+} \left(\frac{\lambda}{|x|^{2s}} |w^{\tau_{n_k}}|^2 + \tau_{n_k}^{2s} h(\tau_{n_k} x) |w^{\tau_{n_k}}|^2 \right. \right. \\ \left. \left. + \frac{\tau_{n_k}^{2s}}{\sqrt{H(\tau_{n_k})}} f(\tau_{n_k} x, \sqrt{H(\tau_{n_k})} w^{\tau_{n_k}}) w^{\tau_{n_k}} \right) dx \right]$$

and

$$H_k(r) = \frac{1}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} |w^{\tau_{n_k}}|^2 dS.$$

Direct calculations yield

$$(93) \quad \mathcal{N}_k(r) := \frac{D_k(r)}{H_k(r)} = \frac{D(\tau_{n_k} r)}{H(\tau_{n_k} r)} = \mathcal{N}(\tau_{n_k} r) \quad \text{for all } r \in (0, 1).$$

From (92), (90), and (91), it follows that, for any fixed $r \in (0, 1)$,

$$(94) \quad D_k(r) \rightarrow \tilde{D}(r),$$

where

$$(95) \quad \tilde{D}(r) = \frac{1}{r^{N-2s}} \left[\int_{B_r^+} t^{1-2s} |\nabla \tilde{w}|^2 dt dx - \kappa_s \int_{B_r^+} \frac{\lambda}{|x|^{2s}} \tilde{w}^2 dx \right] \quad \text{for all } r \in (0, 1).$$

On the other hand, by compactness of the trace embedding (25), we also have

$$(96) \quad H_k(r) \rightarrow \tilde{H}(r) \quad \text{for any fixed } r \in (0, 1),$$

where

$$(97) \quad \tilde{H}(r) = \frac{1}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} \tilde{w}^2 dS.$$

From (32) it follows that $\tilde{D}(r) > -\frac{N-2s}{2} \tilde{H}(r)$ for all $r \in (0, 1)$. Therefore, if, for some $r \in (0, 1)$, $\tilde{H}(r) = 0$ then $\tilde{D}(r) > 0$, and passing to the limit in (93) should give a contradiction with Lemma 3.14. Hence $H_w(r) > 0$ for all $r \in (0, 1)$ and the function

$$\tilde{\mathcal{N}}(r) := \frac{\tilde{D}(r)}{\tilde{H}(r)}$$

is well defined for $r \in (0, 1)$. From (93), (94), (96), and Lemma 3.14, we deduce that

$$(98) \quad \tilde{\mathcal{N}}(r) = \lim_{k \rightarrow \infty} \mathcal{N}(\tau_{n_k} r) = \gamma$$

for all $r \in (0, 1)$. Therefore $\tilde{\mathcal{N}}$ is constant in $(0, 1)$ and hence $\tilde{\mathcal{N}}'(r) = 0$ for any $r \in (0, 1)$. By (88) and Lemma 3.10 with $h \equiv 0$ and $f \equiv 0$, we obtain

$$\left(\int_{S_r^+} t^{1-2s} \left| \frac{\partial \tilde{w}}{\partial \nu} \right|^2 dS \right) \cdot \left(\int_{S_r^+} t^{1-2s} \tilde{w}^2 dS \right) - \left(\int_{S_r^+} t^{1-2s} \tilde{w} \frac{\partial \tilde{w}}{\partial \nu} dS \right)^2 = 0 \quad \text{for all } r \in (0, 1),$$

which implies that \tilde{w} and $\frac{\partial \tilde{w}}{\partial \nu}$ have the same direction as vectors in $L^2(S_r^+; t^{1-2s})$ and hence there exists a function $\eta = \eta(r)$ such that $\frac{\partial \tilde{w}}{\partial \nu}(r, \theta) = \eta(r) \tilde{w}(r, \theta)$ for all $r \in (0, 1)$ and $\theta \in \mathbb{S}_+^N$. After integration we obtain

$$(99) \quad \tilde{w}(r, \theta) = e^{\int_1^r \eta(s) ds} \tilde{w}(1, \theta) = \varphi(r) \psi(\theta), \quad r \in (0, 1), \theta \in \mathbb{S}_+^N,$$

where $\varphi(r) = e^{\int_1^r \eta(s) ds}$ and $\psi(\theta) = \tilde{w}(1, \theta)$. From (88), (99), and Lemma 2.1, it follows that, weakly,

$$\begin{cases} \frac{1}{r^N} (r^{N+1-2s} \varphi')' \theta_1^{1-2s} \psi(\theta) + r^{-1-2s} \varphi(r) \operatorname{div}_{\mathbb{S}^N} (\theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi(\theta)) = 0, \\ -\lim_{\theta_1 \rightarrow 0^+} \theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi(\theta) \cdot \mathbf{e}_1 = \kappa_s \lambda \psi(0, \theta'). \end{cases}$$

Taking r fixed we deduce that ψ is an eigenfunction of the eigenvalue problem (10). If $\mu_{k_0}(\lambda)$ is the corresponding eigenvalue then $\varphi(r)$ solves the equation

$$\frac{1}{r^N} (r^{N+1-2s} \varphi')' - \mu_{k_0}(\lambda) r^{-1-2s} \varphi(r) = 0$$

i.e.

$$\varphi''(r) + \frac{N+1-2s}{r} \varphi' - \frac{\mu_{k_0}(\lambda)}{r^2} \varphi(r) = 0$$

and hence $\varphi(r)$ is of the form

$$\varphi(r) = c_1 r^{\sigma_{k_0}^+} + c_2 r^{\sigma_{k_0}^-}$$

for some $c_1, c_2 \in \mathbb{R}$, where

$$\sigma_{k_0}^+ = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_{k_0}(\lambda)} \quad \text{and} \quad \sigma_{k_0}^- = -\frac{N-2s}{2} - \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_{k_0}(\lambda)}.$$

Since the function $|x|^{\sigma_{k_0}^-} \psi(\frac{x}{|x|}) \notin L^2(B'_1; |x|^{-2s})$ and hence $|z|^{\sigma_{k_0}^-} \psi(\frac{z}{|z|}) \notin H^1(B_1^+; t^{1-2s})$ in virtue of Lemma 2.5, we deduce that $c_2 = 0$ and $\varphi(r) = c_1 r^{\sigma_{k_0}^+}$. Moreover, from $\varphi(1) = 1$, we obtain that $c_1 = 1$ and then

$$(100) \quad \tilde{w}(r, \theta) = r^{\sigma_{k_0}^+} \psi(\theta), \quad \text{for all } r \in (0, 1) \text{ and } \theta \in \mathbb{S}_+^N.$$

It remains to prove part (i). From (100) and the fact that $\int_{\mathbb{S}_+^N} \theta_1^{1-2s} \psi^2(\theta) dS = 1$ it follows that

$$\begin{aligned} \tilde{D}(r) &= \frac{1}{r^{N-2s}} \left[\int_{B_r^+} t^{1-2s} |\nabla \tilde{w}|^2 dt dx - \kappa_s \int_{B_r^+} \frac{\lambda}{|x|^{2s}} \tilde{w}^2 dx \right] \\ &= r^{2s-N} (\sigma_{k_0}^+)^2 \int_0^r t^{N-1-2s+2\sigma_{k_0}^+} dt \\ &\quad + r^{2s-N} \left(\int_0^r t^{N-1-2s+2\sigma_{k_0}^+} dt \right) \left(\int_{\mathbb{S}_+^N} \theta_1^{1-2s} |\nabla_{\mathbb{S}^N} \psi(\theta)|^2 dS - \lambda \kappa_s \int_{\mathbb{S}^{N-1}} |\mathcal{T}\psi(\theta')|^2 dS' \right) \\ &= \frac{(\sigma_{k_0}^+)^2 + \mu_{k_0}(\lambda)}{N-2s+2\sigma_{k_0}^+} r^{2\sigma_{k_0}^+} = \sigma_{k_0}^+ r^{2\sigma_{k_0}^+} \end{aligned}$$

and

$$\tilde{H}(r) = \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \tilde{w}^2(r\theta) dS = r^{2\sigma_{k_0}^+},$$

and hence from (98) it follows that $\gamma = \tilde{\mathcal{N}}(r) = \frac{\tilde{D}(r)}{\tilde{H}(r)} = \sigma_{k_0}^+$. This completes the proof of the lemma. ■

The following lemma describes the behavior of $H(r)$ as $r \rightarrow 0^+$.

Lemma 4.3. *Let w satisfy (34), H be defined in (50), and let $\gamma := \lim_{r \rightarrow 0^+} \mathcal{N}(r)$ as in Lemma 3.14. Then the limit*

$$\lim_{r \rightarrow 0^+} r^{-2\gamma} H(r)$$

exists and it is finite.

PROOF. In view of (75) it is sufficient to prove that the limit exists. By (50), (52), and Lemma 3.14 we have

$$\frac{d}{dr} \frac{H(r)}{r^{2\gamma}} = -2\gamma r^{-2\gamma-1} H(r) + r^{-2\gamma} H'(r) = 2r^{-2\gamma-1} (D(r) - \gamma H(r)) = 2r^{-2\gamma-1} H(r) \int_0^r \mathcal{N}'(\rho) d\rho.$$

Integration over (r, \tilde{R}) yields

$$(101) \quad \frac{H(\tilde{R})}{\tilde{R}^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{\tilde{R}} 2s^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_1(t) dt \right) d\rho + \int_r^{\tilde{R}} 2\rho^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_2(t) dt \right) d\rho$$

where ν_1 and ν_2 are as in (61) and (62). Since, by Schwarz's inequality, $\nu_1 \geq 0$, we have that $\lim_{r \rightarrow 0^+} \int_r^{\tilde{R}} 2\rho^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_1(t) dt \right) d\rho$ exists. On the other hand, by (75), Lemma 3.12, and (73), we deduce that

$$\begin{aligned} \left| \rho^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_2(t) dt \right) \right| &\leq K_1 C_1 \left(C_2 + \frac{N-2s}{2} \right) \rho^{-1} \int_0^\rho \left(t^{-1+\varepsilon} + t^{-1+\frac{2s(p_0-2^*(s))}{p_0}} + g(t) \right) dt \\ &\leq K_1 C_1 \left(C_2 + \frac{N-2s}{2} \right) \rho^{-1} \left(\frac{\rho^\varepsilon}{\varepsilon} + \frac{p_0 \rho^{\frac{2s(p_0-2^*(s))}{p_0}}}{2s(p_0-2^*(s))} + \frac{1}{1-\alpha} \|w\|_{L^p(B'_R)}^{p(1-\alpha)} \rho^{N\left(\frac{\alpha 2^*(s)-2}{2^*(s)} - \frac{p\alpha-2}{p_0}\right)} \right) \end{aligned}$$

for all $\rho \in (0, \tilde{R})$, which proves that $\rho^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_2(t) dt \right) \in L^1(0, \tilde{R})$. Hence both terms at the right hand side of (101) admit a limit as $r \rightarrow 0^+$ thus completing the proof. ■

From Lemma 4.2, the following pointwise estimate for solutions to (1) and (35) can be derived.

Lemma 4.4. *Let w satisfying (35). Then there exist $C_4, C_5 > 0$ and $\bar{r} \in (0, \tilde{R})$ such that*

- (i) $\sup_{S_r^+} |w|^2 \leq \frac{C_4}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} |w(z)|^2 dS$ for every $0 < r < \bar{r}$,
- (ii) $|w(z)| \leq C_5 |z|^\gamma$ for all $z \in B_{\bar{r}}^+$ and in particular $|w(0, x)| \leq C_5 |x|^\gamma$ for all $x \in B_{\bar{r}}^t$, where $\gamma := \lim_{r \rightarrow 0^+} \mathcal{N}(r)$ is as in Lemma 3.14.

PROOF. We first notice that (ii) follows directly from (i) and (75). In order to prove (i), we argue by contradiction and assume that there exists a sequence $\tau_n \rightarrow 0^+$ such that

$$\sup_{\theta \in \mathbb{S}_+^N} \left| w\left(\frac{\tau_n}{2}\theta\right) \right|^2 > nH\left(\frac{\tau_n}{2}\right)$$

with H as in (50), i.e.

$$\sup_{x \in S_{1/2}^+} |w(\tau_n z)|^2 > 2^{N+1-2s} n \int_{S_{1/2}^+} t^{1-2s} w^2(\tau_n z) dS,$$

i.e., defining w^τ as in (80)

$$(102) \quad \sup_{x \in S_{1/2}^+} |w^{\tau_n}(z)|^2 > 2^{N+1-2s} n \int_{S_{1/2}^+} t^{1-2s} |w^{\tau_n}(z)|^2 dS.$$

From Lemma 4.2, along a subsequence τ_{n_k} we have that $w^{\tau_{n_k}} \rightarrow |z|^\gamma \psi\left(\frac{z}{|z|}\right)$ in $C_{\text{loc}}^{0,\alpha}(S_{1/2}^+)$, for some ψ eigenfunction of problem (10), hence passing to the limit in (102) gives rise to a contradiction. ■

We will now prove that $\lim_{r \rightarrow 0^+} r^{-2\gamma} H(r)$ is strictly positive.

Lemma 4.5. *Under the same assumption as in Lemmas 4.3 and 4.4, we have*

$$\lim_{r \rightarrow 0^+} r^{-2\gamma} H(r) > 0.$$

PROOF. For all $k \geq 1$, let ψ_k be as in (21), i.e. ψ_k is a $L^2(\mathbb{S}_+^N; \theta_1^{1-2s})$ -normalized eigenfunction of problem (10) associated to the eigenvalue $\mu_k(\lambda)$ and $\{\psi_k\}_k$ is an orthonormal basis of $L^2(\mathbb{S}_+^N; \theta_1^{1-2s})$. From Lemma 4.2 there exist $j_0, m \in \mathbb{N}$, $j_0, m \geq 1$ such that m is the multiplicity of the eigenvalue $\mu_{j_0}(\lambda) = \mu_{j_0+1}(\lambda) = \dots = \mu_{j_0+m-1}(\lambda)$ and

$$(103) \quad \gamma = \lim_{r \rightarrow 0^+} \mathcal{N}(r) = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_i(\lambda)}, \quad i = j_0, \dots, j_0 + m - 1.$$

Let us expand w as

$$w(z) = w(\tau\theta) = \sum_{k=1}^{\infty} \varphi_k(\tau) \psi_k(\theta)$$

where $\tau = |z| \in (0, R]$, $\theta = z/|z| \in \mathbb{S}_+^N$, and

$$(104) \quad \varphi_k(\tau) = \int_{\mathbb{S}_+^N} \theta_1^{1-2s} w(\tau\theta) \psi_k(\theta) dS.$$

The Parseval identity yields

$$(105) \quad H(\tau) = \int_{\mathbb{S}_+^N} \theta_1^{1-2s} w^2(\tau\theta) dS = \sum_{k=1}^{\infty} \varphi_k^2(\tau), \quad \text{for all } 0 < \tau \leq R.$$

In particular, from (75) and (105) it follows that, for all $k \geq 1$,

$$(106) \quad \varphi_k(\tau) = O(\tau^\gamma) \quad \text{as } \tau \rightarrow 0^+.$$

Equations (35) and (21) imply that, for every k ,

$$-\varphi_k''(\tau) - \frac{N+1-2s}{\tau} \varphi_k'(\tau) + \frac{\mu_k(\lambda)}{\tau^2} \varphi_k(\tau) = \zeta_k(\tau), \quad \text{in } (0, R),$$

where

$$(107) \quad \zeta_k(\tau) = \frac{\kappa_s}{\tau^{2-2s}} \int_{\mathbb{S}^{N-1}} (h(\tau\theta') w(0, \tau\theta') + f(\tau\theta', u(\tau\theta'))) \psi_k(0, \theta') dS'.$$

A direct calculation shows that, for some $c_1^k, c_2^k \in \mathbb{R}$,

$$(108) \quad \varphi_k(\tau) = \tau^{\sigma_k^+} \left(c_1^k + \int_{\tau}^R \frac{t^{-\sigma_k^++1}}{\sigma_k^+ - \sigma_k^-} \zeta_k(t) dt \right) + \tau^{\sigma_k^-} \left(c_2^k + \int_{\tau}^R \frac{t^{-\sigma_k^-+1}}{\sigma_k^- - \sigma_k^+} \zeta_k(t) dt \right),$$

where

$$(109) \quad \sigma_k^+ = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_k(\lambda)} \quad \text{and} \quad \sigma_k^- = -\frac{N-2s}{2} - \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_k(\lambda)}.$$

From (3), (4), Lemma 4.4, (103), and the fact that, in view of (20),

$$2s + (p-2)\gamma = \frac{(N-2s)(2^*(s)-p)}{2} + (p-2)\sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_{j_0}(\lambda)} > 0,$$

we deduce that, for all $i = j_0, \dots, j_0 + m - 1$,

$$(110) \quad \zeta_i(\tau) = O(\tau^{-2+\tilde{\delta}+\sigma_i^+}) \quad \text{as } \tau \rightarrow 0^+,$$

with $\tilde{\delta} = \min\{\varepsilon, 2s + (p-2)\gamma\} > 0$. Consequently, the functions

$$t \mapsto \frac{t^{-\sigma_i^++1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) \quad \text{and} \quad t \mapsto \frac{t^{-\sigma_i^-+1}}{\sigma_i^- - \sigma_i^+} \zeta_i(t)$$

belong to $L^1(0, R)$. Hence

$$\tau^{\sigma_i^+} \left(c_1^i + \int_{\tau}^R \frac{\rho^{-\sigma_i^++1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(\rho) d\rho \right) = o(\tau^{\sigma_i^-}) \quad \text{as } \tau \rightarrow 0^+,$$

and then, by (106), there must be

$$c_2^i = - \int_0^R \frac{t^{-\sigma_i^-+1}}{\sigma_i^- - \sigma_i^+} \zeta_i(t) dt.$$

Using (110), we then deduce that

$$(111) \quad \tau^{\sigma_i^-} \left(c_2^i + \int_{\tau}^R \frac{t^{-\sigma_i^-+1}}{\sigma_i^- - \sigma_i^+} \zeta_i(t) dt \right) = \tau^{\sigma_i^-} \left(\int_0^{\tau} \frac{t^{-\sigma_i^-+1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) dt \right) = O(\tau^{\sigma_i^+ + \bar{\delta}})$$

as $\tau \rightarrow 0^+$. From (108) and (111), we obtain that, for all $i = j_0, \dots, j_0 + m - 1$,

$$(112) \quad \varphi_i(\tau) = \tau^{\sigma_i^+} \left(c_1^i + \int_{\tau}^R \frac{t^{-\sigma_i^++1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) dt + O(\tau^{\bar{\delta}}) \right) \quad \text{as } \tau \rightarrow 0^+.$$

Let us assume by contradiction that $\lim_{\lambda \rightarrow 0^+} \lambda^{-2\gamma} H(\lambda) = 0$. Then, for all $i \in \{j_0, \dots, j_0 + m - 1\}$, (103) and (105) would imply that

$$\lim_{\tau \rightarrow 0^+} \tau^{-\sigma_i^+} \varphi_i(\tau) = 0.$$

Hence, in view of (112),

$$c_1^i + \int_0^R \frac{t^{-\sigma_i^++1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) dt = 0,$$

which, together with (110), implies

$$(113) \quad \tau^{\sigma_i^+} \left(c_1^i + \int_{\tau}^R \frac{t^{-\sigma_i^++1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) dt \right) = \tau^{\sigma_i^+} \int_0^{\tau} \frac{t^{-\sigma_i^++1}}{\sigma_i^- - \sigma_i^+} \zeta_i(t) dt = O(\tau^{\sigma_i^+ + \bar{\delta}})$$

as $\tau \rightarrow 0^+$. Collecting (108), (111), and (113), we conclude that

$$\varphi_i(\tau) = O(\tau^{\sigma_i^+ + \bar{\delta}}) \quad \text{as } \tau \rightarrow 0^+ \quad \text{for every } i \in \{j_0, \dots, j_0 + m - 1\},$$

namely,

$$\sqrt{H(\tau)} (w^{\tau}, \psi)_{L^2(\mathbb{S}_+^N; \theta_1^{1-2s})} = O(\tau^{\gamma + \bar{\delta}}) \quad \text{as } \tau \rightarrow 0^+$$

for every $\psi \in \mathcal{A}_0 = \text{span}\{\psi_i\}_{i=j_0}^{j_0+m-1}$, where \mathcal{A}_0 is the eigenspace of problem (10) associated to the eigenvalue $\mu_{j_0}(\lambda) = \mu_{j_0+1}(\lambda) = \dots = \mu_{j_0+m-1}(\lambda)$. From (76), there exists $C(\bar{\delta}) > 0$ such that $\sqrt{H(\tau)} \geq C(\bar{\delta}) \tau^{\gamma + \frac{\bar{\delta}}{2}}$ for τ small, and therefore

$$(114) \quad (w^{\tau}, \psi)_{L^2(\mathbb{S}_+^N; \theta_1^{1-2s})} = O(\tau^{\frac{\bar{\delta}}{2}}) \quad \text{as } \tau \rightarrow 0^+$$

for every $\psi \in \mathcal{A}_0$. From Lemma 4.2, for every sequence $\tau_n \rightarrow 0^+$, there exist a subsequence $\{\tau_{n_k}\}_{k \in \mathbb{N}}$ and an eigenfunction $\tilde{\psi} \in \mathcal{A}_0$

$$(115) \quad \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \tilde{\psi}^2(\theta) dS = 1 \quad \text{and} \quad w^{\tau_{n_k}} \rightarrow \tilde{\psi} \quad \text{in } L^2(\mathbb{S}_+^N; \theta_1^{1-2s}).$$

From (114) and (115), we infer that

$$0 = \lim_{k \rightarrow +\infty} (w^{\tau_{n_k}}, \tilde{\psi})_{L^2(\mathbb{S}^{N-1})} = \|\tilde{\psi}\|_{L^2(\mathbb{S}_+^N; \theta_1^{1-2s})}^2 = 1,$$

thus reaching a contradiction. ■

We can now completely describe the behavior of solutions to (35) near the singularity, hence proving Theorem 4.1.

PROOF OF THEOREM 4.1. Identity (79) follows from part (i) of Lemma 4.2, thus there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that $\gamma = \lim_{r \rightarrow 0^+} \mathcal{N}(r) = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_{k_0}(\lambda)}$. Let us denote as m the multiplicity of $\mu_{j_0}(\lambda)$ so that, for some $j_0 \in \mathbb{N}$, $j_0 \geq 1$, $j_0 \leq k_0 \leq j_0 + m - 1$, $\mu_{j_0}(\lambda) = \mu_{j_0+1}(\lambda) = \dots = \mu_{j_0+m-1}(\lambda)$ and let $\{\psi_i\}_{i=j_0}^{j_0+m-1}$ be an $L^2(\mathbb{S}_+^N; \theta_1^{1-2s})$ -orthonormal basis for the eigenspace associated to $\mu_{k_0}(\lambda)$.

Let $\{\tau_n\}_{n \in \mathbb{N}} \subset (0, +\infty)$ such that $\lim_{n \rightarrow +\infty} \tau_n = 0$. Then, from part (ii) of Lemma 4.2 and Lemmas 4.3 and 4.5, there exist a subsequence $\{\tau_{n_k}\}_{k \in \mathbb{N}}$ and m real numbers $\beta_{j_0}, \dots, \beta_{j_0+m-1} \in \mathbb{R}$

such that $(\beta_{j_0}, \beta_{j_0+1}, \dots, \beta_{j_0+m-1}) \neq (0, 0, \dots, 0)$ and

$$(116) \quad \tau_{n_k}^{-\gamma} w(\tau_{n_k} \theta) \rightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{0,\alpha}(\mathbb{S}_+^N) \quad \text{as } k \rightarrow +\infty,$$

$$(117) \quad \tau_{n_k}^{-\gamma} w(0, \tau_{n_k} \theta') \rightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(0, \theta') \quad \text{in } C^{1,\alpha}(\mathbb{S}^{N-1}) \quad \text{as } k \rightarrow +\infty,$$

and

$$(118) \quad \tau_{n_k}^{1-\gamma} \nabla_x w(0, \tau_{n_k} \theta') \rightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i (\gamma \psi_i(0, \theta') \theta' + \nabla_{\mathbb{S}^{N-1}} \psi_i(0, \theta')) \quad \text{in } C^{0,\alpha}(\mathbb{S}^{N-1}) \quad \text{as } k \rightarrow +\infty$$

for some $\alpha \in (0, 1)$.

We now prove that the β_i 's depend neither on the sequence $\{\tau_n\}_{n \in \mathbb{N}}$ nor on its subsequence $\{\tau_{n_k}\}_{k \in \mathbb{N}}$.

Defining φ_i and ζ_i as in (104) and (107), from (116) it follows that, for any $i = j_0, \dots, j_0 + m - 1$,

$$(119) \quad \tau_{n_k}^{-\gamma} \varphi_i(\tau_{n_k}) = \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \frac{w(\tau_{n_k} \theta)}{\tau_{n_k}^\gamma} \psi_i(\theta) dS \rightarrow \sum_{j=j_0}^{j_0+m-1} \beta_j \int_{\mathbb{S}_+^N} \theta_1^{1-2s} \psi_j(\theta) \psi_i(\theta) dS = \beta_i$$

as $k \rightarrow +\infty$. As deduced in the proof of Lemma 4.5, for any $i = j_0, \dots, j_0 + m - 1$ and $\tau \in (0, R]$ there holds

$$(120) \quad \begin{aligned} \varphi_i(\tau) &= \tau^{\sigma_i^+} \left(c_1^i + \int_\tau^R \frac{t^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) dt \right) + \tau^{\sigma_i^-} \left(\int_0^\tau \frac{t^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) dt \right) \\ &= \tau^{\sigma_i^+} \left(c_1^i + \int_\tau^R \frac{t^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) dt + O(\tau^\delta) \right) \quad \text{as } \tau \rightarrow 0^+, \end{aligned}$$

for some $c_1^i \in \mathbb{R}$, where σ_i^\pm are defined in (109). Choosing $\tau = R$ in the first line of (120), we obtain

$$c_1^i = R^{-\sigma_i^+} \varphi_i(R) - R^{\sigma_i^- - \sigma_i^+} \int_0^R \frac{s^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(s) ds.$$

Hence (120) yields

$$\tau^{-\gamma} \varphi_i(\tau) \rightarrow R^{-\sigma_i^+} \varphi_i(R) - R^{\sigma_i^- - \sigma_i^+} \int_0^R \frac{t^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) dt + \int_0^R \frac{t^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(t) dt \quad \text{as } \tau \rightarrow 0^+,$$

and therefore from (119) we deduce that

$$\begin{aligned} \beta_i &= R^{-\gamma} \int_{\mathbb{S}_+^N} \theta_1^{1-2s} w(R\theta) \psi_i(\theta) dS \\ &\quad - R^{-2\gamma - N + 2s} \int_0^R \frac{\kappa_s \rho^{\gamma + N - 1}}{2\gamma + N - 2s} \left(\int_{\mathbb{S}^{N-1}} (h(\rho\theta') w(0, \rho\theta') + f(\rho\theta', w(0, \rho\theta'))) \psi_i(0, \theta') dS' \right) d\rho \\ &\quad + \int_0^R \frac{\kappa_s \rho^{2s - \gamma - 1}}{2\gamma + N - 2s} \left(\int_{\mathbb{S}^{N-1}} (h(\rho\theta') w(0, \rho\theta') + f(\rho\theta', w(0, \rho\theta'))) \psi_i(0, \theta') dS' \right) d\rho. \end{aligned}$$

In particular the β_i 's depend neither on the sequence $\{\tau_n\}_{n \in \mathbb{N}}$ nor on its subsequence $\{\tau_{n_k}\}_{k \in \mathbb{N}}$, thus implying that the convergences in (116) and (117), and (118) actually hold as $\tau \rightarrow 0^+$ and proving the theorem. \square

5. PROOF OF THEOREM 1.1

Let $\mathcal{D}^{s,2}(\Omega)$ denote the completion on $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$. Simple density arguments show that (7) is equivalent to

$$(121) \quad (u, \varphi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_\Omega \left(\frac{\lambda}{|x|^{2s}} u(x) + h(x) u(x) + f(x, u(x)) \right) \varphi(x) dx, \quad \text{for all } \varphi \in \mathcal{D}^{s,2}(\Omega).$$

Since $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, we can let $\mathcal{H}(u) \in \mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ such that

$$\int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla \mathcal{H}(u) \cdot \nabla \varphi \, dt \, dx = 0, \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}_+^{N+1}),$$

and $\mathcal{H}(u) = u$ on \mathbb{R}^N identified with $\partial\mathbb{R}_+^{N+1}$, i.e. $\mathcal{H}(u)$ weakly satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla \mathcal{H}(u)) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ \mathcal{H}(u) = u, & \text{on } \partial\mathbb{R}_+^{N+1} = \{0\} \times \mathbb{R}^N. \end{cases}$$

From [4] we have that

$$\int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla \mathcal{H}(u) \cdot \nabla \tilde{\varphi} \, dt \, dx = \kappa_s(u, \tilde{\varphi})_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{for all } \tilde{\varphi} \in \mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s}),$$

where

$$\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)},$$

i.e.

$$-\lim_{t \rightarrow 0^+} t^{1-2s} \frac{\partial \mathcal{H}(u)}{\partial t} = \kappa_s(-\Delta)^s u(x),$$

in a weak sense. Therefore $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ weakly solves (1) in Ω in the sense of (7) if and only if its extension $w = \mathcal{H}(u)$ satisfies

$$(122) \quad \begin{cases} \operatorname{div}(t^{1-2s} \nabla w) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ w = u, & \text{on } \mathbb{R}^N, \\ -\lim_{t \rightarrow 0^+} t^{1-2s} \frac{\partial w}{\partial t}(t, x) = \kappa_s \left(\frac{\lambda}{|x|^{2s}} w + hw + f(x, w) \right), & \text{on } \Omega, \end{cases}$$

in a weak sense, i.e. if for all $\tilde{\varphi} \in \mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ such that $x \mapsto \tilde{\varphi}(0, x) \in \mathcal{D}^{s,2}(\Omega)$ we have

$$(123) \quad \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla w \cdot \nabla \tilde{\varphi} \, dt \, dx = \kappa_s \int_{\Omega} \left(\frac{\lambda}{|x|^{2s}} w + hw + f(x, w) \right) \tilde{\varphi} \, dx.$$

Since $\mathcal{D}^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s}) \hookrightarrow H^1(B_R^+; t^{1-2s})$ for all $R > 0$, the result follows from Theorem 4.1. \square

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